Can Relativistic MOND Theory Resolve Both the Dark Matter and Dark Energy Paradigms?

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In this letter, we point out the possibility of a unified description of the roles of dark matter and dark energy within the framework a recently proposed relativistic MOND theory. In addition to the known successes of this model in explaining the rotation curves of galaxies, we suggest that the homogeneous part of the dynamical scalar field could play the role of dark energy. A viable cosmological model is proposed and its dynamics is studied.

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Current astronomical observations on cosmological scales (CMB anisotropy [1, 2, 3], supernovae [4, 5, 6] and SDSS[7]) reveal that our universe is spatially flat, with about two thirds of the energy content resulting from what is referred to as dark energy. This energy has a negative pressure to account for the accelerated expansion of the universe. While, on smaller, galactic scales, the rotational curves of galaxies strongly indicate that the biggest contributions to their mass density arises from non-luminous matter, which has given rise to speculations on the existence of dark matter[8].

On the other hand, it is also interesting to inquire if we can solve the current observational puzzles, both dark matter and dark energy, by modifying Einstein gravity in spite of its many successes, in particular, the solar system tests. MOND theory is a striking modification of the Newtonian laws of motion and explains the rotational curves with amazing accuracy[9] without introducing any dark matter. After some initial problems a consistent relativistic extension of the MOND theory has recently been proposed by Bekenstein[10]. In that extension, general relativity has been modified by including two scalar fields (one dynamical while the other is non-dynamical), one vector field and a conformal coupling of the scalar field to the metric tensor (hence the acronym TeVeS associated with it). In addition, the theory contains one arbitrary function whose form is dictated in part by its primary purpose, i.e., to address the data on the rotational curves of galaxies, or equivalently, the dark matter problem. The remarkable features of this theory are that it can explain the galaxy rotational curves without introducing dark matter (just as its non-relativistic counterpart, MOND theory) and at the same time reduce to general relativity in the appropriate limits. The price one pays is the additional complication of the whole system: two scalar fields, one vector field, gravitational field and the conformal coupling.

A natural question to ask is if the new framework also works for cosmology? Can it help tackling the dark energy problem? We will show in this paper that under appropriate conditions, the scalar field could mimic the behavior of the dark energy models[11, 12, 13, 14, 15] currently in vogue. Thus, the TeVeS theory has the potential, to naturally solve the cosmic puzzles currently explained by dark energy and dark matter without invoking either of them explicitly.

We begin with a brief outline of cosmology in the TeVeS framework. Following Bekenstein[10], the action for the whole system could be written as

$$S = S_g + S_s + S_v + S_m \tag{1}$$

where

$$S_g = (16\pi G)^{-1}\int g^{\alpha\beta}R_{\alpha\beta}\sqrt{-g}d^4x \tag{2}$$

$$S_s = -\frac{1}{2}\int\left[\sigma^2 h^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + \frac{1}{2}G\ell^{-2}\sigma^4F(kG\sigma^2)\right]\sqrt{-g}d^4x \tag{3}$$

$$\begin{aligned} S_v &= -\frac{K}{32\pi G}\int\left[g^{\alpha\beta}g^{\mu\nu}\mathfrak{U}_{[\alpha,\mu]}\mathfrak{U}_{[\beta,\nu]}\right. \\ &\quad \left.-2(\lambda/K)(g^{\mu\nu}\mathfrak{U}_\mu\mathfrak{U}_\nu+1)\right]\sqrt{-g}d^4x, \end{aligned} \tag{4}$$

$$S_m = \int\mathcal{L}(\tilde{g}_{\mu\nu}, f^\alpha, f^\alpha_{|\mu}, \cdots)\sqrt{-\tilde{g}}d^4x \tag{5}$$

are the actions for gravity, scalar field, vector field and luminous matter respectively. In the above, $\tilde{g}_{\alpha\beta} = e^{-2\phi}(g_{\alpha\beta}+\mathfrak{U}_\alpha\mathfrak{U}_\beta) - e^{2\phi}\mathfrak{U}_\alpha\mathfrak{U}_\beta$ is the metric tensor in real world and $g_{\mu\nu}$ is the Einstein metric tensor. $\mathfrak{U}_\alpha$ is a time-like 4-vector field, ϕ and σ are respectively the dynamical the non-dynamical scalar fields, F is a free function to be specified by dynamics while k and K are two positive dimensionless parameters. Varying the action, one can arrive at the following equations of motion for the vector, scalar and gravitational fields [10]:

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$$\begin{aligned} K & \left(\mathfrak{U}^{[\alpha;\beta]}{}_{;\beta} + \mathfrak{U}^\alpha \mathfrak{U}_\gamma \mathfrak{U}^{[\gamma;\beta]}{}_{;\beta}\right) + 8\pi G\sigma^2 \left[\mathfrak{U}^\beta \phi_{,\beta}\, g^{\alpha\gamma} \phi_{,\gamma} + \mathfrak{U}^\alpha (\mathfrak{U}^\beta \phi_{,\beta})^2\right] \\ = & \ 8\pi G(1 - e^{-4\phi}) \left[g^{\alpha\mu} \mathfrak{U}^\beta \tilde{T}_{\mu\beta} + \mathfrak{U}^\alpha \mathfrak{U}^\beta \mathfrak{U}^\gamma \tilde{T}_{\gamma\beta}\right] \end{aligned} \tag{6}$$

$$-\mu F(\mu) - \frac{1}{2}\mu^2 F'(\mu) = k\ell^2 h^{\alpha\beta} \phi_{,\alpha}\phi_{,\beta} \tag{7}$$

$$\left[\mu\left(k\ell^2 h^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right) h^{\alpha\beta}\phi_{,\alpha}\right]_{;\beta} = kG\left[g^{\alpha\beta} + (1 + e^{-4\phi})\mathfrak{U}^\alpha\mathfrak{U}^\beta\right]\tilde{T}_{\alpha\beta}. \tag{8}$$

$$G_{\alpha\beta} = 8\pi G\left[\tilde{T}_{\alpha\beta} + (1 - e^{-4\phi})\mathfrak{U}^\mu \tilde{T}_{\mu(\alpha}\mathfrak{U}_{\beta)} + \tau_{\alpha\beta}\right] + \Theta_{\alpha\beta} \tag{9}$$

where

$$\begin{aligned} \tau_{\alpha\beta} & \equiv \sigma^2\left[\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\, g_{\alpha\beta} - \mathfrak{U}^\mu\phi_{,\mu}\left(\mathfrak{U}_{(\alpha}\phi_{,\beta)} - \frac{1}{2}\mathfrak{U}^\nu\phi_{,\nu}\, g_{\alpha\beta}\right)\right] \\ & - \frac{1}{4}G\ell^{-2}\sigma^4 F(kG\sigma^2) g_{\alpha\beta} \end{aligned} \tag{10}$$

$$\Theta_{\alpha\beta} \equiv K\left(g^{\mu\nu}\mathfrak{U}_{[\mu,\alpha]}\mathfrak{U}_{[\nu,\beta]} - \frac{1}{4}g^{\sigma\tau}g^{\mu\nu}\mathfrak{U}_{[\sigma,\mu]}\mathfrak{U}_{[\tau,\nu]}\, g_{\alpha\beta}\right) - \lambda\mathfrak{U}_\alpha\mathfrak{U}_\beta \tag{11}$$

$\tilde{T}_{\mu\nu}$ is the energy momentum tensor of ordinary matter in the physical coordinate system. $h^{\alpha\beta} = g^{\alpha\beta} - \mathfrak{U}^\alpha\mathfrak{U}^\beta$ and $\mu = kG\sigma^2$.

In a cosmological setting, the symmetries of the FRW universe will simplify the above equations considerably. It is worth recalling here an underlying assumption in such applications. One interprets the fields in the problem, say the scalar field, as consisting of both a spatially homogenous (time dependent) part and an inhomogenous one. On cosmological scales the spatially inhomogenous part may be neglected in the first approximation while at galactic scales, the inhomogenous part plays a prominent role and the homogenous part is negligible (quasi-static approximation). Henceforth, we will restrict ourselves to cosmological scales, more specifically to the flat universe case, and consider the line element:

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}(\tilde{t})^2 d\mathbf{x}^2 \tag{12}$$

where $d\tilde{t} = e^\phi dt$ and $\tilde{a} = e^{-\phi}a$. The vector field in this case could be chosen as $\mathfrak{U}^\mu = \delta^\mu_t$ and the scalar field depends only on time t. Thus, the Einstein equations become

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_M + \rho_\phi) \tag{13}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_M + \rho_\phi + 3p_M + 3p_\phi) \tag{14}$$

where $\rho_M = \tilde{\rho}e^{-2\phi}$, $p_M = \tilde{p}e^{-2\phi}$ are respectively the density and pressure of luminous matter in Einstein's picture. ρ_ϕ and p_ϕ are the effective energy density and pressure of the scalar field ϕ, defined as:

$$\begin{aligned} \rho_\phi &= \frac{\mu\dot{\phi}^2}{kG} + \frac{\mu^2}{4\ell^2k^2G}F(\mu) \\ p_\phi &= \frac{\mu\dot{\phi}^2}{kG} - \frac{\mu^2}{4\ell^2k^2G}F(\mu) \end{aligned} \tag{15}$$

where μ relates to $\dot{\phi}$ via the σ equation (Eq.7):

$$\mu F(\mu) + \frac{1}{2}\mu^2 F'(\mu) = 2k\ell^2\dot{\phi}^2 \tag{16}$$

The equation of motion of the scalar field ϕ in a FRW background is given by

$$\mu\ddot{\phi} + (3H\mu + \dot{\mu})\dot{\phi} + \frac{kG}{2}(\rho_M + 3p_M) = 0 \quad (17)$$

where $H = \dot{a}/a$ and dot denotes the derivative with respect to t. It is worth nothing that Eq.(17) is consistent with the condition $\dot{\phi} \sim 0$ because the matter density ρ_M decreases to zero at late times. Therefore, the "slow roll" approximation for the scalar field ϕ is consistent only at late times. If we assume that $\dot{\phi}$ varies slowly and $e^{-2\phi} \sim 1$, then we can consider the two energy components to be approximately adiabatic, and energy momentum conservation leads to

$$\frac{d\rho_i}{dt} = -3\frac{\dot{a}}{a}(\rho_i + p_i) \quad (18)$$

where the subscript i denotes M and ϕ. To get a closed system of equations, we also need to specify the equation of state of the energy/matter content. This equation of state is specified by $w = p/\rho$, which is 0 for non-relativistic matter. While for the scalar field, this equation of state is given by its equation of motion, i.e.,Eq.(17) which can be expressed as,

$$w_\phi = \frac{4\ell^2 k\mu\dot{\phi}^2 - \mu^2 F(\mu)}{4\ell^2 k\mu\dot{\phi}^2 + \mu^2 F(\mu)} \quad (19)$$

In the "slow roll" regime, $\dot{\phi} \sim \ddot{\phi} \sim 0$ and $w_\phi \sim -1$. It is worth noting that from Eq.(19), the equation of state could approach -1 from either greater than -1 (quintessence case) or less than -1(phantom case) if we choose the form of $F(\mu)$ appropriately. In this paper, we focus on the case with $w_\phi \geq -1$. Eqs.(13-14) are expressed in terms of the scale factor in the Einstein picture. In terms of the physical picture scale factor we have the relation:

$$\begin{aligned} \frac{\dot{\tilde{a}}}{\tilde{a}} &= e^{-\phi}\left(\frac{\dot{a}}{a} - \dot{\phi}\right) \quad (20) \\ \frac{\ddot{\tilde{a}}}{\tilde{a}} &= e^{-2\phi}\left(\frac{\ddot{a}}{a} - 3\frac{\dot{a}}{a}\dot{\phi} + 2\dot{\phi}^2 - \ddot{\phi}\right) \end{aligned}$$

It is obvious that in the "slow roll" regime, the accelerated or deccelerated expansion of Universe is equivalently reflected by $\ddot{a}$ with $e^{-2\phi} \sim 1$. We will now discuss a viable cosmological model.

From the above discussion, it is clear now that we will get the evolution of the whole dynamical system if the form of $F(\mu)$ is specified. However, at present there are no theoretical arguments in favor of any specific choice, thereby providing a lot of freedom in this regard. The constraints on the form of $F(\mu)$ are phenomenological in nature motivated by the condition that the correct physics is obtained. In this section, we will consider the the choice of the form of $F(\mu)$ that will lead to an acceptable cosmology in addition to accommodating the MOND theory and Newton's Law.

From Eq.(15), the positive energy condition, $\rho_\phi \geq 0$ and the non-vanishing of ρ_ϕ as $\dot{\phi} \to 0$ may restrict the freedom in the choice of the form of $F(\mu)$. Simplicity motivates the choice $\mu^2 F(\mu) = const + p(\mu)$ with $p(\mu)$ a function of μ. The constant term will guarantee a non-vanishing energy density as $\dot{\phi} \to 0$. Thus, we specify the form of $F(\mu)$ as

$$F(\mu) = \frac{3}{8}\frac{\mu(4 + 2\mu - 4\mu^2 + \mu^3) + 2\ln[(1-\mu)^2]}{\mu^2} + \frac{\alpha}{\mu^2} \quad (21)$$

where α is a dimensionless constant. When $\alpha = 0$, the above reduces to the form in[10]. Then, from Eq.(16), we can obtain the relation between μ and $\dot{\phi}$ as

$$\frac{3}{2}(1 + \mu - 3\mu^2 + \mu^3 - \frac{1}{1-\mu}) = 4k\ell^2\dot{\phi}^2 \quad (22)$$

For a consistent cosmology, we require $2 \leq \mu < \infty$. Note also that $\dot{\phi} = 0$ when $\mu = 2$, for which the energy density Eq.(15) becomes $\rho_\phi = \frac{\alpha}{4\ell^2 k^2 G}$ and the equation of state reduces to, $w_\phi = -1$ at late times, which corresponds to a de Sitter phase. This not only provides an acceptable cosmological state but also ensures the consistency of our previous assumption $\dot{\phi} \sim 0$. Next, we show that such a phase corresponds to a dynamical de Sitter attractor[16]. To do this, we introduce the dimensionless variables $x = \phi$, $z = t_0\dot{\phi}$ and $N = \ln a$ with t_0 a constant of dimension t. We can rewrite the Eqs.(13-18) in terms of x, z and N and linearize them around the critical point $(x, z) = (x, 0)$, then, we arrive at the following system of equations

$$\begin{aligned} \frac{dx}{dN} &= \frac{z}{t_0\sqrt{2\pi\alpha/3k^2\ell^2}} \quad (23) \\ \frac{dz}{dN} &= -3z \end{aligned}$$

It is easy to see that the eigenvalues of the coefficients matrix of Eq.(23) are $(-3, 0)$ which indicates that the critical point is stable, i.e. a dynamical attractor. One comes to the same conclusion by noting that $\dot{\phi} = 0$ leads to a minimum of the energy density ρ_ϕ. In Figs. (1) and (2), we plot the numerical results for the dynamical system defined above. Note that our intention here is merely to illustrate the possibility of a consistent cosmology and not to fit the exact observational data. So, for convenience, we have set the parameter $\alpha = 0.01$ and the rest to unity. Since the attractor corresponds to all ϕ with $\dot{\phi} = 0$, we can choose $\phi \sim 0$ so that $e^{-2\phi} \sim 1$, which is indicated by the previous discussion. In our numerical analyses, we chose $\phi = 0.00001$ and the initial $\dot{\phi}$ from 0.001 to 0.005 with an increment of 0.001.

Summarizing, in this paper, we have studied the cosmological implications of the relativistic MOND theory

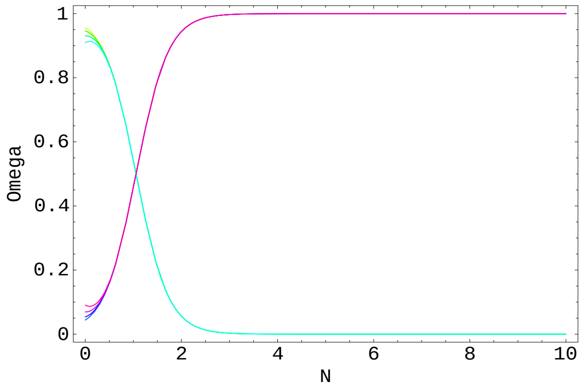

FIG. 1: The evolution of cosmic parameters for matter(indigo curve) and ϕ field (pink curve).

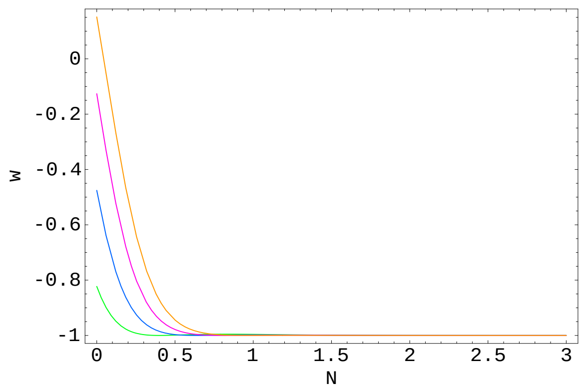

FIG. 2: The evolution of the equation of state of ϕ field for different initial $\dot{\phi}$.

and interpreted the dynamical scalar field as dark energy. We constructed a viable cosmological model, in which the dynamical system has a late time de Sitter behavior.

On the other hand, the framework of the relativistic MOND theory involves an arbitrary function in addition to many parameters as well as auxiliary fields. This not only provides much freedom in matching the theory to observations, but also adds uncertainties. Clearly, they must be constrained from further phenomenological investigations of the testable predications of the theory. It would be interesting to see if the model can be made compatible with all the astronomical data with a unique choice for the arbitrary function. Are there any theoretical constraints on this function?

The analysis in this paper is an initial attempt in this direction. The choice of the function $F(\mu)$, though without any deep theoretical basis, phenomenologically explains both dark matter and dark energy in the same framework. It remains to be checked in future how this form of the function $F(\mu)$ can be embedded in the larger context of an appropriate function that explains all the data.

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