Geometric In-Flight Calibration of JunoCam Using Stars of a Single Swath

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Abstract

JunoCam has been calibrated pre-flight in the laboratories of MSSS. Since JunoCam has been designed as an education and outreach camera, a science-level calibration hasn't been an objective. Formal geometric calibration results are a little more accurate than to two pixels. Residual errors have been documented. They provide an option for a refinement of the thus far published formal geometry. Accurate geometric calibration is a necessary ingredient for an accurate registration of the RGB color channels, and for accurate measurements of the geometry and dynamics of Jupiter's top clouds. Accurate RGB color registration is necessary for any highly resolved inference of compositional properties of the clouds.

During cruise JunoCam took several swaths of the sky background. Several of those cruise images have been made available by MSSS. They provide independent and additional ways for geometric camera calibration. One approach is using stars. They provide an absolute scale of the image geometry. In some of the swaths, it's easily possible to identify blips as stars up to a visual magnitude of about 5.0. This provides geometric data for several dozens of stars per swath.

The centroids of some blips can be determined with an accuracy of less than one pixel. After discaring some outliers, experimental RMS calculations of the considered (between 60 and 70) stars, however, indicate only a mean accuracy of the blip centroids of about 3.5 pixels.

This article defines a family of geometric camera models formally, and calculates their partial derivatives with respect to the parameters of the model. These derivatives are used to determine the best-fit values of these parameters by applying a quasi-Newton method to the sum of the squared distances between each considered BSC star position predicted by the model and the according blip centroid in the respective swath.

The obtained rotation per subframe for the selected swath is -0.077835, with a systematic relative accuracy of about $2 \cdot 10^{-5}$, or 0.1 pixels for 40 subframes. The obtained horizontal position of the optical axis is near 814 with a systematic absolute accuracy of about 0.7 pixels. The obtained Brownian radial parameter K_1 varied between about $-3.1 \cdot 10^{-8}$ and $3.7 \cdot 10^{-8}$, depending on the vertical position of the optical axis, if only K_1 has been allowed. Allowing for K_2 returned less stable values and modified K_1 considerably. Results for the vertical position of the optical axis have been somewhat confusing, since they differ about 120 pixels from the nominal assumption; this might, however, be a result of the biased input comprising only data of the red readout region. [1]

[1]This document was typeset with LaTeX.

This article has been published as part of a post at `unmannedspaceflight.com`.

Contents

1 Introduction

Juno's Education and Outreach camera JunoCam took more than 100 images between launch and the end of the year 2014. Malin Space Science Systems, San Diego, CA, USA (MSSS) made available most of them in a PDS-like pre-release on an MSSS web sever [12].

Besides the images taken by JunoCam during the Earth flyby (EFB) in October 2013 [11], the image collection comprises several test images, and swaths of the sky intended to observe stars or zodiacal light. Some of the swaths have been taken with up to 80 time delay integration (TDI) steps. They allow indentification of several stars up to a visual magnitude of at least 5.0.

Some of the images are lossy compressed. Particularly for the red filter band several swaths of good quality are available. Some of the swaths show more than 80 blips which can be associated with Bright Star Catalog (BSC) [3] stars.

The BSC data, as provided in the bsc5p star catalog, allow a calculation of the pointing vectors of stars in Euclidean coordinates corresponding to the J2000.0 system.

A geometric camera model derived from the calibration provided by MSSS [5] (method), and [8] (laboratory results), together with some estimates, allows a simulation of the BSC stars.

Once three stars are identifid in the swath, one technique is determining the pointing vectors of the three stars, i.e. the centroids of the blips, with respect to the first framelet and the estimated camera model. If regular, these three vectors define the base of a 3-dimensional vector space. The vectors of the same three stars within the J2000.0 system provide a second base of a 3-dimensional vector space. Identifying both bases as identical, by multiplying the 3×3-matrices formed by the vectors to AB^{-1}, provides a linear transformation between the J2000.0 frame and the estimated camera frame. This transformation can then be used to simulate the BSC stars within the estimated camera frame. Since $AB^{-1} =: (t_1, t_2, t_3)$ isn't perfectly orthonormal, in general, it's better to correct it by replacing e.g. the second and third column of the three-column matrix in two steps by applying the vector product $t'_3 := t_1 \times t_2$, and $t'_2 := t'_3 \times t_1$. Provided the initial estimate has been sufficiently good, the simulated stars are sufficiently close to the blips of the respective swath to allow an identification of several more blips with BSC stars.

Between the centroid of the observed blip and the simulated star position, there is some distance. The distance d_i for identified blip i can e.g. be measuered in pixels. The square of the distances can be summed up to

$$f(p_1, \ldots, p_m) = \sum_{i=1}^{n} {d_i}^2(p_1, \ldots, p_m), \tag{1}$$

with n the number of identified blips, and m the number of parameters p_i of the considered camera model. The goal is finding parameters $p_1, \ldots, p_m$ which minimize $f(p_1, \ldots, p_m)$.

Assuming $f(p_1, \ldots, p_m)$ to be differentiable twice on a parameter domain considered for optimization, the gradient of f is zero at the minimum of f. Despite not all zeroes of ∇f being local minima, this suggests an application of the Newton method to find a zero for ∇f. Besides the gradient, the Newton method requires the second derivative of f, which means the Hessian, for f with more than one parameter.

Section 2 provides a general framework of geometric camera models, and the resulting patterns occuring in the according calculus. The considered geometric camera models consist of sequences of transformations applied to unit vectors in $\mathbf{R}^3$. These transformations comprise rotations, projections and non-linear distortions. They can be expressed as nested functions. Deriving nested functions invokes the generalized chain rule. Since only some of the parameters of the transformations are themselves transformations, some general simplifications applicable to all considered camera models are possible.

Section 3 defines the considered geometric transformation steps, and their partial derivatives.

Section 4 discusses the results, and informally, some possible model extensions.

Some of the purposes of this article are providing a mathematical basis for the development of a calibration software for JunoCam, and to publish some of the obtained calibration results. It may be put into the context of [8, subsection 6.4], goal 3: "Provide data to the amateur image processing community and encourage them to produce a variety of products".

Most of the mathematical basics can be found in textbooks like [4] (geometry, calculus) or [7] (algebra).

2 Geometric Framework and Calculus Patterns

2.1 Framework for Geometric Camera Models

2.1.1 Notational Variations for the Square Error of the Model

For $1 \leq i \leq n$, let

$$C_i := \begin{pmatrix} c_{1,i} \\ c_{2,i} \end{pmatrix} \tag{2}$$

be the position of the centroid of blip i. Let

$$\begin{array}{rcl} T : \mathbf{R}^{m+3} & \rightarrow & \mathbf{R}^2, \\ (p_1, \ldots, p_m, v_{1,i}, v_{2,i}, v_{3,i}) & \mapsto & T(p_1, \ldots, p_m, v_{1,i}, v_{2,i}, v_{3,i}) \end{array} \tag{3}$$

transform a vector

$$V_i := \begin{pmatrix} v_{1,i} \\ v_{2,i} \\ v_{3,i} \end{pmatrix} \tag{4}$$

to a point

$$X_i := \begin{pmatrix} x_{1,i} \\ x_{2,i} \end{pmatrix} := T(P, V_i) =: \begin{pmatrix} t_1(P, V_i) \\ t_2(P, V_i) \end{pmatrix}, \tag{5}$$

considering the parameters

$$P := \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix}. \tag{6}$$

Then in equation (1), the squared distances ${d_i}^2(p_1, \ldots, p_m)$ are given as

$${d_i}^2(p_1, \ldots, p_m) = (x_{1,i} - c_{1,i})^2 + (x_{2,i} - c_{2,i})^2, \tag{7}$$

or more explicitely as

$$\begin{array}{rcl} {d_i}^2(p_1, \ldots, p_m) & = & (t_1(p_1, \ldots, p_m, v_{1,i}, v_{2,i}, v_{3,i}) - c_{1,i})^2 \\ & + & (t_2(p_1, \ldots, p_m, v_{1,i}, v_{2,i}, v_{3,i}) - c_{2,i})^2. \end{array} \tag{8}$$

This can be written as a scalar product

$${d_i}^2(p_1, \ldots, p_m) = (x_{1,i} - c_{1,i}, x_{2,i} - c_{2,i}) \cdot \begin{pmatrix} x_{1,i} - c_{1,i} \\ x_{2,i} - c_{2,i} \end{pmatrix}, \tag{9}$$

by equations (2), (5), and (6) as

$${d_i}^2(P) = (X_i - C_i)^\top \cdot (X_i - C_i), \tag{10}$$

or as

$${d_i}^2(P) = (T(P, V_i) - C_i)^\top \cdot (T(P, V_i) - C_i). \tag{11}$$

Defining

$$\|V\|^2 := V^\top \cdot V \tag{12}$$

for arbitrary column vectors V, simplify equations (11) and (10) to

$${d_i}^2(P) = \|X_i - C_i\| = \|T(P, V_i) - C_i)\|^2. \tag{13}$$

This allows writig equation (1) as

$$f(P) = \sum_{i=1}^{n} \|T(P, V_i) - C_i)\|^2. \tag{14}$$

2.1.2 Composing Transformations

Define $V_{1,i}$ as

$$V_{1,i} := \begin{pmatrix} p_1 \\ \vdots \\ p_m \\ v_{1,i} \\ v_{2,i} \\ v_{3,i} \end{pmatrix}. \tag{15}$$

For all individual transformation steps T_j the full parameter set P is provided.

For some $h \in \mathbf{N}$ and an appropriate $1 \le j_1 \le h$, each vector $V_{1,i} \in \mathbf{R}^{m+3}$ pointing to a star i, the transformations of the considered camera models first, for $1 \le j < j_1$, step by step transform $V_{j,i}$ to other vectors $V_{j+1,i} \in \mathbf{R}^{m+3}$, then a transformation projects the vector $V_{j_1,i} \in \mathbf{R}^{m+3}$ to a vector $V_{j_1+1,i} \in \mathbf{R}^{m+2}$, then, for $j_1 < j \le h$, from vectors in $V_{j,i} \in \mathbf{R}^{m+2}$ to vectors $V_{j+1,i} \in \mathbf{R}^{m+2}$.

More formally: For $1 \le j < j_1$, define the individual transformations as

$$\begin{aligned} T_j : \mathbf{R}^{m+3} &\rightarrow \mathbf{R}^{m+3}, \\ \begin{pmatrix} p_1 \\ \vdots \\ p_m \\ v_{j,1,i} \\ v_{j,2,i} \\ v_{j,3,i} \end{pmatrix} &\mapsto \begin{pmatrix} p_1 \\ \vdots \\ p_m \\ t_{1,j}(p_1, \dots, p_m, v_{j,1,i}, v_{j,2,i}, v_{j,3,i}) \\ t_{2,j}(p_1, \dots, p_m, v_{j,1,i}, v_{j,2,i}, v_{j,3,i}) \\ t_{3,j}(p_1, \dots, p_m, v_{j,1,i}, v_{j,2,i}, v_{j,3,i}) \end{pmatrix}. \end{aligned} \tag{16}$$

For $j = j_1$, define the transformation as

$$\begin{aligned} T_j : \mathbf{R}^{m+3} &\rightarrow \mathbf{R}^{m+2}, \\ \begin{pmatrix} p_1 \\ \vdots \\ p_m \\ v_{j,1,i} \\ v_{j,2,i} \\ v_{j,3,i} \end{pmatrix} &\mapsto \begin{pmatrix} p_1 \\ \vdots \\ p_m \\ t_{1,j}(p_1, \dots, p_m, v_{j,1,i}, v_{j,2,i}, v_{j,3,i}) \\ t_{2,j}(p_1, \dots, p_m, v_{j,1,i}, v_{j,2,i}, v_{j,3,i}) \end{pmatrix}. \end{aligned} \tag{17}$$

For $j_1 < j \leq h$, define the individual transformations as

$$
\begin{array}{rcl}
T_j : \mathbf{R}^{m+2} & \to & \mathbf{R}^{m+2}, \\
\begin{pmatrix} p_1 \\ \vdots \\ p_m \\ v_{j,1,i} \\ v_{j,2,i} \end{pmatrix} & \mapsto & \begin{pmatrix} p_1 \\ \vdots \\ p_m \\ t_{1,j}(p_1, \ldots, p_m, v_{j,1,i}, v_{j,2,i}, v_{j,3,i}) \\ t_{2,j}(p_1, \ldots, p_m, v_{j,1,i}, v_{j,2,i}, v_{j,3,i}) \end{pmatrix}.
\end{array} \tag{18}
$$

Based on these individual transformation steps, write the composed transformation T as

$$
T(P, V_i) = \pi_{m+1,m+2}(T_h(T_{h-1}(\ldots T_1(V_{1,i}) \ldots))), \tag{19}
$$

with $\pi_{m+1,m+2}$ projecting to the last two components. For the sake of notational simplicity, the final projection step will be neglected, formally resulting in some overloading of $T(P, V_i)$, allowing for the equation

$$
T(P, V_i) = T_h(T_{h-1}(\ldots T_1(V_{1,i}) \ldots)). \tag{20}
$$

Equation (14) can then be written as

$$
f(P) = \sum_{i=1}^{n} \| T_h(T_{h-1}(\ldots T_2(T_1(V_{1,i}) \ldots)) - C_i) \|^2. \tag{21}
$$

2.2 Newton Method

For a twice derivable function

$$
\begin{array}{rcl}
f : \mathbf{R}^m & \to & \mathbf{R}, \\
(p_1, \ldots, p_m) & \mapsto & f(p_1, \ldots, p_m),
\end{array} \tag{22}
$$

the gradient (first derivative)

$$
\nabla f(p_1, \ldots, p_m) = \begin{pmatrix} \frac{\partial f(p_1, \ldots, p_m)}{\partial p_1} \\ \vdots \\ \frac{\partial f(p_1, \ldots, p_m)}{\partial p_m} \end{pmatrix} \tag{23}
$$

equals zero for the minima of f.

For a sufficiently well-behaved ∇f, the Newton method approximates a zero of ∇f by iterating

$$
\begin{pmatrix} p_{1,k+1} \\ \vdots \\ p_{m,k+1} \end{pmatrix} := \begin{pmatrix} p_{1,k} \\ \vdots \\ p_{m,k} \end{pmatrix} - (Hf(p_{1,k}, \ldots, p_{m,k}))^{-1} \cdot \nabla f(p_{1,k}, \ldots, p_{m,k}), \tag{24}
$$

with the Hessian

$$Hf(p_1,\ldots,p_m) = \begin{pmatrix} \frac{\partial^2 f(p_1,\ldots,p_m)}{\partial^2 p_1} & \cdots & \frac{\partial^2 f(p_1,\ldots,p_m)}{\partial p_1 \partial p_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(p_1,\ldots,p_m)}{\partial p_m \partial p_1} & \cdots & \frac{\partial^2 f(p_1,\ldots,p_m)}{\partial^2 p_m} \end{pmatrix}, \tag{25}$$

and starting with a suitable $(p_{1,0},\ldots,p_{m,0})$.

In many cases, the double partial derivatives needed for the Hessian can be implemented as quotients of differences, resulting in a quasi-Newton method.

2.3 Partial Derivatives

2.3.1 Deriving the Square Error

Let $1 \leq k \leq m$, and f as defined in equation (1). Then

$$\frac{\partial f(p_1,\ldots,p_m)}{\partial p_k} = 2 \cdot \sum_{i=1}^{n} d_i(p_1,\ldots,p_m) \cdot \frac{\partial d_i(p_1,\ldots,p_m)}{\partial p_k}. \tag{26}$$

By equation (7),

$$d_i(p_1,\ldots,p_m) = \sqrt{(x_{1,i} - c_{1,i})^2 + (x_{2,i} - c_{2,i})^2}. \tag{27}$$

More explicitely, with equation (8),

$$\begin{aligned} d_i(p_1,\ldots,p_m) &= [(t_1(p_1,\ldots,p_m,v_{1,i},v_{2,i},v_{3,i}) - c_{1,i})^2 \\ &+ (t_2(p_1,\ldots,p_m,v_{1,i},v_{2,i},v_{3,i}) - c_{2,i})^2]^{\frac{1}{2}}. \end{aligned} \tag{28}$$

Deriving d_i partially with respect to p_k results in

$$\begin{aligned} \frac{\partial d_i(P)}{\partial p_k} &= \left[(t_1(P,V_i) - c_{1,i})^2 + (t_2(P,V_i) - c_{2,i})^2\right]^{-\frac{1}{2}} \\ &\cdot \left[(t_1(P,V_i) - c_{1,i}) \cdot \frac{\partial t_1(P,V_i)}{\partial p_k} + (t_2(P,V_i) - c_{2,i}) \cdot \frac{\partial t_2(P,V_i)}{\partial p_k}\right]. \end{aligned} \tag{29}$$

With equation (8), and using the definition of the scalar product of two vectors this can be written as

$$\frac{\partial d_i(P)}{\partial p_k} = \frac{1}{d_i(P)} \cdot (t_1(P,V_i) - c_{1,i}, t_2(P,V_i) - c_{2,i}) \cdot \begin{pmatrix} \frac{\partial t_1(P,V_i)}{\partial p_k} \\ \frac{\partial t_2(P,V_i)}{\partial p_k} \end{pmatrix}, \tag{30}$$

and by using equations (2) and (5) as

$$\frac{\partial d_i(P)}{\partial p_k} = \frac{1}{d_i(P)} \cdot (T(P,V_i) - C_i)^\top \cdot \begin{pmatrix} \frac{\partial t_1(P,V_i)}{\partial p_k} \\ \frac{\partial t_2(P,V_i)}{\partial p_k} \end{pmatrix}. \tag{31}$$

When defining the component-wise partial derivative of a vector of functions, this can be written in a shorter form as

$$\frac{\partial d_i(P)}{\partial p_k} \quad = \quad \frac{1}{d_i(P)} \cdot (T(P,V_i) - C_i)^\top \cdot \frac{\partial T(P,V_i)}{\partial p_k}. \tag{32}$$

Applying this to equation (26) results in

$$\frac{\partial f(P)}{\partial p_k} = 2 \cdot \sum_{i=1}^{n} d_i(P) \cdot \frac{1}{d_i(P)} \cdot (T(P,V_i) - C_i)^\top \cdot \frac{\partial T(P,V_i)}{\partial p_k}, \tag{33}$$

which simplifies to

$$\frac{\partial f(P)}{\partial p_k} = 2 \cdot \sum_{i=1}^{n} (T(P,V_i) - C_i)^\top \cdot \frac{\partial T(P,V_i)}{\partial p_k}. \tag{34}$$

Note the relationship to equation (14).

By $\nabla \sum f = \sum \nabla f$, equations (23) and (34) can be combined to

$$\nabla f(p_1,\ldots,p_m) = 2 \cdot \sum_{i=1}^{n} \begin{pmatrix} (T(P,V_i) - C_i)^\top \cdot \frac{\partial T(P,V_i)}{\partial p_1} \\ \vdots \\ (T(P,V_i) - C_i)^\top \cdot \frac{\partial T(P,V_i)}{\partial p_m} \end{pmatrix}, \tag{35}$$

or after writing the scalar product as sum of products,

$$\nabla f(p_1,\ldots,p_m) = 2 \cdot \sum_{i=1}^{n} \begin{pmatrix} (t_1(P,V_i) - c_{1,i}) \cdot \frac{\partial t_1(P,V_i)}{\partial p_1} + (t_2(P,V_i) - c_{2,i}) \cdot \frac{\partial t_2(P,V_i)}{\partial p_1} \\ \vdots \\ (t_1(P,V_i) - c_{1,i}) \cdot \frac{\partial t_1(P,V_i)}{\partial p_m} + (t_2(P,V_i) - c_{2,i}) \cdot \frac{\partial t_2(P,V_i)}{\partial p_m} \end{pmatrix}. \tag{36}$$

Again applying the linearity of the ∇-operator results in

$$\nabla f(P) = 2 \cdot \sum_{i=1}^{n} \left[(t_1(P,V_i) - c_{1,i}) \cdot \nabla t_1(P,V_i) + (t_2(P,V_i) - c_{2,i}) \cdot \nabla t_2(P,V_i) \right]. \tag{37}$$

In a more general notation,

$$\nabla f(P) = 2 \cdot \sum_{i=1}^{n} \sum_{\alpha=1}^{2} (t_\alpha(P,V_i) - c_{\alpha,i}) \cdot \nabla t_\alpha(P,V_i). \tag{38}$$

Exploitig the formal analogy of equation (34) between derivatives of one function with those of vectors of functions for the second derivative returns

$$\frac{\partial^2 f(P)}{\partial p_k \partial p_l} = 2 \cdot \sum_{i=1}^{n} \left[\left(\frac{\partial T(P,V_i)}{\partial p_l} \right)^\top \cdot \frac{\partial T(P,V_i)}{\partial p_k} + (T(P,V_i) - C_i)^\top \cdot \frac{\partial^2 T(P,V_i)}{\partial p_k \partial p_l} \right]. \tag{39}$$

2.3.2 Deriving Composed Transformations

Assume $T(P, V_i)$ being composed as in equation (20). Then the partial derivative with respect to p_k is

$$\frac{\partial T(P, V_i)}{\partial p_k} = \frac{\partial T_h(T_{h-1}(\dots T_1(V_{1,i})\dots))}{\partial p_k}. \tag{40}$$

The partial derivatives can be composed to the Jacobian

$$\frac{\partial T(P, V_i)}{\partial P} = \frac{\partial T_h(T_{h-1}(\dots T_1(V_{1,i})\dots))}{\partial P}. \tag{41}$$

Application of the generalized chain rule returns

$$\frac{\partial T(P, V_i)}{\partial P} = \frac{\partial T_h(T_{h-1}(\dots T_1(V_{1,i})\dots))}{\partial T_{h-1}(\dots T_1(V_{1,i})\dots)} \cdot \frac{\partial T_{h-1}(\dots T_1(V_{1,i})\dots))}{\partial T_{h-2}(\dots T_1(V_{1,i})\dots)} \cdots \frac{\partial T_1(V_{1,i})}{\partial P}, \tag{42}$$

or written as a formal product,

$$\frac{\partial T(P, V_i)}{\partial P} = \prod_{j=0}^{h-1} \frac{\partial T_{h-j}(T_{h-j-1}(\dots T_1(V_{1,i})\dots))}{\partial T_{h-j-1}(\dots T_1(V_{1,i})\dots)}, \tag{43}$$

when defining

$$T_0(V_{1,i}) := P. \tag{44}$$

The domains as well as the codomains of the T_j are column vectors, so the partial derivatives

$$\mathbf{J}T_{h-j}(T_{h-j-1}(\dots T_1(V_{1,i})\dots)) := \frac{\partial T_{h-j}(T_{h-j-1}(\dots T_1(V_{1,i})\dots))}{\partial T_{h-j-1}(\dots T_1(V_{1,i})\dots)} \tag{45}$$

are Jacobian matrices. Note, that

$$\frac{\partial p_k}{\partial p_l} = \begin{cases} 1, \text{ for } k = l \\ 0, \text{ else,} \end{cases} \tag{46}$$

and

$$\frac{\partial T_j(p, \tilde{P})}{\partial p} = 0, \tag{47}$$

if essentially

$$T_j(p, \tilde{P}) = T_j(\tilde{P}). \tag{48}$$

The second derivative of the composition

$$\frac{\partial^2 T(P, V_i)}{\partial p_k \partial p_l}$$

of transformations is obtained by applying the product rule to equation (43).

$$\begin{aligned} \tfrac{\partial^2 T(P,V_i)}{\partial P^2} = \textstyle\sum_{j=0}^{n-1} & \left[\textstyle\prod_{q=0}^{j-1} \tfrac{\partial T_{h-q}(T_{h-q-1}(\dots T_1(V_{1,i})\dots))}{\partial T_{h-q-1}(\dots T_1(V_{1,i})\dots)}\right] \\ & \cdot \tfrac{\partial^2 T_{h-j}(T_{h-j-1}(\dots T_1(V_{1,i})\dots))}{\partial (T_{h-j-1}(\dots T_1(V_{1,i})\dots))^2} \\ & \cdot \left[\textstyle\prod_{q=j+1}^{n-1} \tfrac{\partial T_{h-q}(T_{h-q-1}(\dots T_1(V_{1,i})\dots))}{\partial T_{h-q-1}(\dots T_1(V_{1,i})\dots)}\right], \end{aligned} \tag{49}$$

with empty products defined as 1.

3 Geometric Transformations and Partial Derivatives

3.1 Initial Linear Transformation

3.1.1 Definition

Assume the initial estimate of the transformation from the J2000.0 frame to the camera frame of the first subframe of the considered swath being defined via multiplying the matrix

$$B_1 := \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} \tag{50}$$

from the left to column vectors $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3$, usually of amount $\sqrt{x^2+y^2+z^2} = 1$.

The matrix B should define a right-handed orthonormal transformation, i.e. a rotation as an element of the (multiplicative) Lie-group $SO(3)$, embedded into the linear group $GL_3(\mathbf{R})$. But the constraints aren't a strict requirement for optimization purposes.

Define the transformation T_1 by

$$\begin{aligned} T_1 : \mathbf{R}^{m+3} &\rightarrow \mathbf{R}^{m+3}, \\ \begin{pmatrix} P \\ X \end{pmatrix} &\mapsto \begin{pmatrix} I & 0 \\ 0 & B_1 \end{pmatrix} \cdot \begin{pmatrix} P \\ X \end{pmatrix} = \begin{pmatrix} P \\ B_1 \cdot X \end{pmatrix}, \end{aligned} \tag{51}$$

in block matrix notation with

$$P = \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix},$$

and

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

the unit matrix of the appropriate size.

3.1.2 Partial Derivatives

Since T_1 linear in all parameters, the Jacobi matrix

$$\mathbf{J}T_1 \begin{pmatrix} P \\ X \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B_1 \end{pmatrix} \tag{52}$$

is a constant.

The second derivative is zero.

3.2 Rotation Around z-Axis

3.2.1 Definition

Let $p_1 \in \mathbf{R}$ be an angle (in radians). Multiply the matrix block

$$B_2 := \begin{pmatrix} \cos p_1 & -\sin p_1 & 0 \\ \sin p_1 & \cos p_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{53}$$

from the left to column vectors $X \in \mathbf{R}^3$ to define the linear transformation

$$\begin{array}{rcl} T_2 : \mathbf{R}^{m+3} & \rightarrow & \mathbf{R}^{m+3}, \\ \begin{pmatrix} P \\ X \end{pmatrix} & \mapsto & \begin{pmatrix} I & 0 \\ 0 & B_2 \end{pmatrix} \cdot \begin{pmatrix} P \\ X \end{pmatrix} = \begin{pmatrix} P \\ B_2 \cdot X \end{pmatrix}. \end{array} \tag{54}$$

3.2.2 Partial Derivatives

Since

$$\frac{\partial B_2}{\partial p_1} = \begin{pmatrix} -\sin p_1 & -\cos p_1 & 0 \\ \cos p_1 & -\sin p_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{55}$$

the column of the Jacobi matrix composed of the derivatives with respect to p_1 is

$$\begin{pmatrix} E_1 \\ \begin{pmatrix} -\sin p_1 & -\cos p_1 & 0 \\ \cos p_1 & -\sin p_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot X \end{pmatrix}, \tag{56}$$

with

$$E_i := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{57}$$

the i-th unit vector of appropriate size, with entry 1 in the i-th row, else 0.

The derivative of the matrix block $B_2 \cdot X$ with respect to X is the matrix block $\frac{\partial B_2 \cdot X}{\partial X} = B_2$ of the Jacobi matrix.

According to equations (46) and (47), the entries of the remaining Jacobi matrix are the same as those of the unit matrix of the same size.

Hence

$$\mathbf{J}T_2\begin{pmatrix} P \\ X \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \frac{\partial B_2}{\partial p_1}\cdot X & \mathbf{0} & B_2 \end{pmatrix}, \tag{58}$$

with entries $\mathbf{0}$ representing vector or matrix blocks of appropriate sizes, with all entries 0.

Some second derivative of B_2:

$$\frac{\partial^2 B_2}{\partial p_1^2} = \begin{pmatrix} -\cos p_1 & \sin p_1 & 0 \\ -\sin p_1 & -\cos p_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{59}$$

3.3 Rotation Around y-Axis

3.3.1 Definition

Let $p_2 \in \mathbf{R}$ be an angle (in radians). Multiply the matrix block

$$B_3 := \begin{pmatrix} \cos p_2 & 0 & -\sin p_2 \\ 0 & 1 & 0 \\ \sin p_2 & 0 & \cos p_2 \end{pmatrix}. \tag{60}$$

from the left to column vectors $X \in \mathbf{R}^3$ to define the linear transformation

$$\begin{array}{rcl} T_3 : \mathbf{R}^{m+3} & \longrightarrow & \mathbf{R}^{m+3}, \\ \begin{pmatrix} P \\ X \end{pmatrix} & \mapsto & \begin{pmatrix} I & 0 \\ 0 & B_3 \end{pmatrix} \cdot \begin{pmatrix} P \\ X \end{pmatrix} = \begin{pmatrix} P \\ B_3 \cdot X \end{pmatrix}. \end{array} \tag{61}$$

3.3.2 Partial Derivatives

Since

$$\frac{\partial B_3}{\partial p_2} = \begin{pmatrix} -\sin p_2 & 0 & -\cos p_2 \\ 0 & 0 & 0 \\ \cos p_2 & 0 & -\sin p_2 \end{pmatrix}, \tag{62}$$

the column of the Jacobi matrix composed of the derivatives with respect to p_2 is

$$\begin{pmatrix} E_2 \\ \begin{pmatrix} -\sin p_2 & 0 & -\cos p_2 \\ 0 & 0 & 0 \\ \cos p_2 & 0 & -\sin p_2 \end{pmatrix} \cdot X \end{pmatrix}. \tag{63}$$

The derivative of the matrix block $B_3 \cdot X$ with respect to X is the matrix block $\frac{\partial B_3 \cdot X}{\partial X} = B_3$ of the Jacobi matrix.

The entries of the remaining Jacobi matrix are the same as those of the unit matrix of the same size.

Hence

$$\mathbf{J}T_3\begin{pmatrix} P \\ X \end{pmatrix} = \begin{pmatrix} 1 & 0 & \mathbf{0} & \mathbf{0} \\ 0 & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \frac{\partial B_3}{\partial p_2} \cdot X & \mathbf{0} & B_3 \end{pmatrix}. \tag{64}$$

Some second derivative of B_3:

$$\frac{\partial^2 B_3}{\partial p_2^2} = \begin{pmatrix} -\cos p_2 & 0 & \sin p_2 \\ 0 & 0 & 0 \\ -\sin p_2 & 0 & -\cos p_2 \end{pmatrix}. \tag{65}$$

3.4 Rotation Around x-Axis

3.4.1 Definition

Let $p_3 \in \mathbf{R}$ be an angle (in radians). Multiply the matrix block

$$B_4 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos p_3 & -\sin p_3 \\ 0 & \sin p_3 & \cos p_3 \end{pmatrix}. \tag{66}$$

from the left to column vectors $X \in \mathbf{R}^3$ to define the linear transformation

$$\begin{aligned} T_4 : \mathbf{R}^{m+3} &\rightarrow \mathbf{R}^{m+3}, \\ \begin{pmatrix} P \\ X \end{pmatrix} &\mapsto \begin{pmatrix} I & 0 \\ 0 & B_4 \end{pmatrix} \cdot \begin{pmatrix} P \\ X \end{pmatrix} = \begin{pmatrix} P \\ B_4 \cdot X \end{pmatrix}. \end{aligned} \tag{67}$$

3.4.2 Partial Derivatives

Since

$$\frac{\partial B_4}{\partial p_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin p_3 & -\cos p_3 \\ 0 & \cos p_3 & -\sin p_3 \end{pmatrix}. \tag{68}$$

the column of the Jacobi matrix composed of the derivatives with respect to p_3 is

$$\begin{pmatrix} E_3 \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin p_3 & -\cos p_3 \\ 0 & \cos p_3 & -\sin p_3 \end{pmatrix} \cdot X \end{pmatrix}. \tag{69}$$

The derivative of the matrix block $B_4 \cdot X$ with respect to X is the matrix block $\frac{\partial B_4 \cdot X}{\partial X} = B_4$ of the Jacobi matrix.

The entries of the remaining Jacobi matrix are the same as those of the unit matrix of the same size.

Hence

$$\mathbf{J}T_4 \begin{pmatrix} P \\ X \end{pmatrix} = \begin{pmatrix} I_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \frac{\partial B_4}{\partial p_3} \cdot X & \mathbf{0} & B_4 \end{pmatrix}, \tag{70}$$

with $I_i \in \mathbf{M_R}(i,i)$, the unit matrix block of size $i \times i$.

Some second derivative of B_4:

$$\frac{\partial^2 B_4}{\partial p_3^2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\cos p_3 & \sin p_3 \\ 0 & -\sin p_3 & -\cos p_3 \end{pmatrix}. \tag{71}$$

3.5 Additional Rotation of s-th Subframe Around x-Axis

3.5.1 Definition

Within a swath the camera rotates around the x-axis in steps of a fixed angle $p_4 \in \mathbf{R}$ (in radians) per subframe $s \in \mathbf{N_0}$.

Multiply the matrix block

$$B_5 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos sp_4 & -\sin sp_4 \\ 0 & \sin sp_4 & \cos sp_4 \end{pmatrix}. \tag{72}$$

from the left to column vectors $X \in \mathbf{R}^3$ to define the linear transformation

$$\begin{aligned} T_5 : \mathbf{R}^{m+3} &\longrightarrow \mathbf{R}^{m+3}, \\ \begin{pmatrix} P \\ X \end{pmatrix} &\mapsto \begin{pmatrix} I & 0 \\ 0 & B_5 \end{pmatrix} \cdot \begin{pmatrix} P \\ X \end{pmatrix} = \begin{pmatrix} P \\ B_5 \cdot X \end{pmatrix}. \end{aligned} \tag{73}$$

3.5.2 Partial Derivatives

Since

$$\frac{\partial B_5}{\partial p_4} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -s\sin sp_4 & -s\cos sp_4 \\ 0 & s\cos sp_4 & -s\sin sp_4 \end{pmatrix}. \tag{74}$$

the column of the Jacobi matrix composed of the derivatives with respect to p_4 is

$$\begin{pmatrix} E_4 \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -s\sin sp_4 & -s\cos sp_4 \\ 0 & s\cos sp_4 & -s\sin sp_4 \end{pmatrix} \cdot X \end{pmatrix}. \tag{75}$$

The derivative of the matrix block $B_5 \cdot X$ with respect to X is the matrix block $\frac{\partial B_5 \cdot X}{\partial X} = B_5$ of the Jacobi matrix.

The entries of the remaining Jacobi matrix are the same as those of the unit matrix of the same size.

Hence

$$\mathbf{J}T_5 \begin{pmatrix} P \\ X \end{pmatrix} = \begin{pmatrix} I_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \frac{\partial B_5}{\partial p_4} \cdot X & \mathbf{0} & B_5 \end{pmatrix}. \tag{76}$$

Some second derivative of B_5:

$$\frac{\partial^2 B_5}{\partial p_4^2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin sp_4 - s\cos sp_4 & -\cos sp_4 + s\sin sp_4 \\ 0 & \cos sp_4 - s\sin sp_4 & -\sin sp_4 - s\cos sp_4 \end{pmatrix}. \tag{77}$$

Derivatives with respect to s aren't considered relevant.

3.6 Projection by Pinhole Camera Model

3.6.1 Definition

For the scale factor p_5, and the optical center $\begin{pmatrix} p_6 \\ p_7 \end{pmatrix}$ define the map

$$\begin{aligned} U_6 : \mathbf{R} \times \mathbf{R} \times \mathbf{R}^- &\to \mathbf{R}^2 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\mapsto -\tfrac{p_5}{z} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p_6 \\ p_7 \end{pmatrix}, \end{aligned} \tag{78}$$

with $\mathbf{R}^- := \{z \in \mathbf{R} \mid z < 0\}$.

Use U_6 to define the (partial) transformation

$$\begin{aligned} T_6 : \mathbf{R}^{m+3} &\to \mathbf{R}^{m+2} \\ \begin{pmatrix} P \\ X \end{pmatrix} &\mapsto \begin{pmatrix} P \\ U_6(X) \end{pmatrix}, \text{ if } U_6(X) \text{ defined.} \end{aligned} \tag{79}$$

The map T_6 is undefined for parameters which induce parameters outside the domain of U_6. To cover the full domain of T_6, an error value should be returned whenever U_6 is undefined. Projections of vectors from behind the camera are to be avoided.

3.6.2 Partial Derivatives

With

$$\frac{\partial U_6}{\partial p_5} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{z} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \tag{80}$$

$$\frac{\partial U_6}{\partial p_6} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{81}$$

$$\frac{\partial U_6}{\partial p_7}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{82}$$

$$\frac{\partial U_6}{\partial x}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{p_5}{z}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{83}$$

$$\frac{\partial U_6}{\partial y}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{p_5}{z}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{84}$$

and

$$\frac{\partial U_6}{\partial z}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{p_5}{z^2}\cdot\begin{pmatrix} x \\ y \end{pmatrix}, \tag{85}$$

the total derivative

$$\mathbf{J}T_6\begin{pmatrix} P \\ X \end{pmatrix} = \begin{pmatrix} I_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 1 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{x}{z} & 1 & 0 & \mathbf{0} & -\frac{p_5}{z} & 0 & \frac{p_5 x}{z^2} \\ \mathbf{0} & \frac{y}{z} & 0 & 1 & \mathbf{0} & 0 & -\frac{p_5}{z} & \frac{p_5 y}{z^2} \end{pmatrix}, \tag{86}$$

if $z < 0$.

3.7 Radial Distortion by Taylor Series

3.7.1 Definition

For $i \in \mathbf{N_0}$, let $p_{8,i} \in \mathbf{R}$. Usually $p_{8,0} = 1$. For the optical center $\begin{pmatrix} p_6 \\ p_7 \end{pmatrix}$ define the map

$$\begin{array}{rcl} U_7 : \mathbf{R}^2 & \to & \mathbf{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} (x-p_6)\cdot\sum_{i=0}^{\infty} p_{8,i} r^i \\ (y-p_7)\cdot\sum_{i=0}^{\infty} p_{8,i} r^i \end{pmatrix} + \begin{pmatrix} p_6 \\ p_7 \end{pmatrix}, \end{array} \tag{87}$$

with

$$r := r(x,y,p_6,p_7) := ((x-p_6)^2 + (y-p_7)^2)^{\frac{1}{2}} \tag{88}$$

Use U_7 to define the transformation

$$\begin{array}{rcl} T_7 : \mathbf{R}^{m+2} & \to & \mathbf{R}^{m+2} \\ \begin{pmatrix} P \\ x \\ y \end{pmatrix} & \mapsto & \begin{pmatrix} P \\ U_7(X) \end{pmatrix}. \end{array} \tag{89}$$

The underlying idea dates back to Brook Taylor [14].

3.7.2 Partial Derivatives

For $i \in \mathbf{N_0}$,

$$\frac{\partial U_7}{\partial p_{8,i}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x-p_6)\cdot((x-p_6)^2+(y-p_7)^2)^{\frac{i}{2}} \\ (y-p_7)\cdot((x-p_6)^2+(y-p_7)^2)^{\frac{i}{2}} \end{pmatrix} = \begin{pmatrix} (x-p_6)\cdot r^i \\ (y-p_7)\cdot r^i \end{pmatrix}. \tag{90}$$

Let $r \neq 0$.

$$\frac{\partial U_7}{\partial p_6} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\left[\sum_{i=0}^{\infty} p_{8,i} r^i\right] - (x-p_6)^2 \cdot \sum_{i=1}^{\infty} i p_{8,i} r^{i-2} \\ -(x-p_6)(y-p_7)\cdot \sum_{i=1}^{\infty} i p_{8,i} r^{i-2} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{91}$$

$$\frac{\partial U_7}{\partial p_7} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -(x-p_6)(y-p_7)\cdot \sum_{i=1}^{\infty} i p_{8,i} r^{i-2} \\ -\left[\sum_{i=0}^{\infty} p_{8,i} r^i\right] - (y-p_7)^2 \cdot \sum_{i=1}^{\infty} i p_{8,i} r^{i-2} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{92}$$

$$\frac{\partial U_7}{\partial x} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left[\sum_{i=0}^{\infty} p_{8,i} r^i\right] + (x-p_6)^2 \cdot \sum_{i=1}^{\infty} i p_{8,i} r^{i-2} \\ (x-p_6)(y-p_7)\cdot \sum_{i=1}^{\infty} i p_{8,i} r^{i-2} \end{pmatrix}. \tag{93}$$

$$\frac{\partial U_7}{\partial y} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x-p_6)(y-p_7)\cdot \sum_{i=1}^{\infty} i p_{8,i} r^{i-2} \\ \left[\sum_{i=0}^{\infty} p_{8,i} r^i\right] + (y-p_7)^2 \cdot \sum_{i=1}^{\infty} i p_{8,i} r^{i-2} \end{pmatrix}. \tag{94}$$

The case $r = 0$ requires separate treatment. This specific consideration is necessary in equation (87) only for the respective summand with $i = 1$, hence for

$$(x-p_6)\cdot p_{8,1}\cdot \sqrt{(x-p_6)^2+(y-p_7)^2},$$

and for

$$(y-p_7)\cdot p_{8,1}\cdot \sqrt{(x-p_6)^2+(y-p_7)^2}.$$

The partial derivatives of these summands can be set to zero at $r = 0$.

Based on the partial derivatives, the following formal Jacobi matrix describes the total derivative of T_7:

$$\mathbf{J}T_7 \begin{pmatrix} P \\ x \\ y \end{pmatrix} = \begin{pmatrix} I_5 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 1 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\partial U_7}{\partial p_6} & \frac{\partial U_7}{\partial p_7} & \frac{\partial U_7}{\partial p_{8,0}} & \cdots & \frac{\partial U_7}{\partial p_{8,\infty}} & \mathbf{0} & \frac{\partial U_7}{\partial x} & \frac{\partial U_7}{\partial x} \end{pmatrix}. \tag{95}$$

In real-world applications $p_{8,\infty}$ is to be replaced by a suitable $p_{8,\omega}$, with $\omega < \infty$. Note, that

$$\begin{pmatrix} \frac{\partial U_7}{\partial x} & \frac{\partial U_7}{\partial x} \end{pmatrix}$$

is a 2×2-matrix block.

Some second derivatives of U_7: For $i \in \mathbf{N_0}$,

$$\frac{\partial^2 U_7}{\partial p_{8,i} \partial p_6} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -r^i - (x-p_6)^2 \cdot i \cdot r^{i-2} \\ -(x-p_6) \cdot (y-p_7) \cdot i \cdot r^{i-2} \end{pmatrix}, \tag{96}$$

and

$$\frac{\partial^2 U_7}{\partial p_{8,i} \partial p_7} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -(x-p_6) \cdot (y-p_7) \cdot i \cdot r^{i-2} \\ -r^i - (y-p_7)^2 \cdot i \cdot r^{i-2} \end{pmatrix}. \tag{97}$$

3.8 Radial Brown-Conrady Distortion

The radial [13] Brown-Conrady model [6] and in its later version [2] is the same as the radial distortion by a Taylor series, except that it allows only for $p_{8,i} \neq 0$, if $i \equiv 0 \mod 2$.

3.9 TDI

3.9.1 Definition

Assuming a projected star moving linearly in a 1st order approximation during exposure, the centroid of a blip representing this star can be assumed to be displaced along the y-axis with (TDI-steps - 1)/2. Formally,

$$\begin{aligned} U_9 : \mathbf{R}^2 &\rightarrow \mathbf{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1-w}{2} \end{pmatrix}, \end{aligned} \tag{98}$$

with a constant $w \in \mathbf{R}$ representing the number of TDI steps.

Use U_9 to define the transformation

$$\begin{aligned} T_9 : \mathbf{R}^{m+2} &\rightarrow \mathbf{R}^{m+2} \\ \begin{pmatrix} P \\ x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} P \\ x \\ y + \frac{1-w}{2} \end{pmatrix}. \end{aligned} \tag{99}$$

3.9.2 Partial Derivatives

$$\mathbf{J} T_9 \begin{pmatrix} P \\ x \\ y \end{pmatrix} = I. \tag{100}$$

3.9.3 Limitations

Higher-order effects of TDI (time delay integration) on the blip centroids are skipped in this article.

4 Results and Discussion

4.1 Test Runs

Appendix A lists detailed results of four calibration runs over the same data set.

The four test runs vary in the number of radial Brownian parameters, i.e. whether K2 is added, and in the assumption of whether or not the center of the optical axis can be assumed to be at $p_7 = y = 600$.

4.1.1 Obtaining Input Data

The input data, consisting of blip centroids with assigned BSC stars, have been inferred from image `JNCE_2014038_00R117_V01`, and from a preliminary simulation of the same image using a revised version of the BSC5p star catalog [3]. The simulation used camera parameters as provided by the Junocam paper [8], i.e. a first radial Brownian coefficient $K_1 = -3.839251 \cdot 10^{-8}$, and a FOV of about 58° on a 1600×1200-pixels CCD chip. The optical axis has been assumed at $(822, 600)$. Subframes are counted in the geometrical order of the swath from bottom to top, starting with 0. This order is not necessarily the same as the subframes have been taken by the camera. A full rotation has been estimated by some try and error to correspond to about 81 subframes. A method, how to obtain an estimate of the initial rotation of subframe 0 is described above, in the Introduction.

4.1.2 Camera Spin Around x-Axis in Radians Per Subframe

The parameter p_4 describing the rotation around the x-axis per subframe, varies between $p_4 = -0.077834135585$ (assuming constant $p_7 = 600.0$, but free K_2), and $p_4 = -0.077835602103$ (assuming free p_7, and constant $K_2 = 0$). The relative systematic "error" between these two values is

$$\frac{-0.077835602103}{-0.077834135585} - 1 \approx 1.884 \cdot 10^{-5}.$$

This error corresponds to a discrepancy of about 0.096 pixels between the first and the last row in a swath of height $40 \cdot 128 = 5120$ pixels. A full rotation corresponds to about 80.724 subframes.

This result indicates, that the spin of the probe – measured in subframes – can be determined by a single 40-subframes swath with a (systematic) accuracy of about 0.2 pixels per revolution.

4.1.3 Horizontal Position of Optical Axis

The parameter p_6 describing the horizontal pixel position of the optical center in an image 1648 pixels wide, varies between $p_6 = 813.607125541612$ (assuming constant $p_7 = 600.0$, but free K_2), and $p_6 = 814.276331289716$. The absolute error in pixels between these two values is

$$814.276331289716 - 813.607125541612 = 0.669205748104.$$

4.1.4 Brownian K1

Models allowing only for the Brownian radial parameter K_1, returned values $p_{8,2} = -3.118341654035 \cdot 10^{-8}$ for constant $p_7 = 600.0$, and $p_{8,2} = -3.687397612268 \cdot 10^{-8}$ for free $p_7 = 480.496273868451$. This is in moderate agreement with the laboratory value of $K_1 = -3.839251 \cdot 10^{-8}$.

4.1.5 Allowing Brownian K2

Models including the Brownian radial parameter K_2, returned values $p_{8,2} = -6.040598817509 \cdot 10^{-8}$, $p_{8,4} = 3.967324089476 \cdot 10^{-14}$ for constant $p_7 = 600.0$, and $p_{8,2} = -7.157963248833 \cdot 10^{-8}$, $p_{8,4} = 5.017409363776 \cdot 10^{-14}$ for free $p_7 = 487.662207828404$. Those values are prone to larger errors, since K_2 depends largely on the distortion near the left and right margins of the images. The number of available data points might be too small to allow for statistically reliable values.

4.1.6 Vertical Position of Optical Axis

Models allowing for a free choice of the y-position of the optical axis returned values of about 480 (only K_1 free) and 487 (K_1 and K_2 free). Those values are similar. But they differ considerably from the assumed position 600. Three possible interpretations appear straightforward:

- The assumption of the optical center being near $y = 600$ needs to be corrected to about $y \approx 480$,
- the assumption of the red readout region starting at $y = 306$ needs to be adjusted to $y \approx 426$, or
- the results obtained by a single swath containing only data of the red readout region is heavily biased.

4.1.7 Rotation of Subframe 0 relative to J2000.0

The results depend on the other parameters, due to the asymmetry induced by the position of the red readout region, the only considered. An example is

$$\begin{pmatrix} 0.2506242196 & 0.8307643699 & 0.4859616074 \\ -0.6347046579 & -0.2466465099 & 0.7355383620 \\ 0.7309839244 & -0.4989950509 & 0.4720430427 \end{pmatrix}$$

for free optical axis and only one radial Brownian parameter K_1.

4.2 (Informal) Candidates for Model Extensions

Various effects in addition to the investigated camera model might be considered.

4.2.1 Selected methodological effects

1. Bias due to restriction to red filter readout region. Adding swaths of other filters would reduce the bias.

2. Statistics for 8 or 9 parameters with 68 included blips leaves non-negligible statistical uncertainty. Considering several swaths, and more sensitive blip detection are straightforward approaches.

3. Considering the accuracy of each individual blip centroid as a weight would modify the result.

4. SPICE data and well-known camera data from laboratory tests could be used to reduce the number of free parameters.

5. The centroid calculation software may be improvable, to provide better input data.

4.2.2 Selected camera-intrinsic effects

1. For a swath with 80 TDI steps, the image of a star on the CCD may cross parts of the green filter, a small gap, and parts of the red filter, before it is read out as a short streak. Depending on the CCD position within each TDI step, the streak varies in brightness. This variability within one streak (blip) depends on the position of the blip.

2. The sensitivity of the CCD varies with each pixel.

3. The incidence angle of star light on the CCD chip varies with position (flat field correction).

4. Background stray light modifies centroid calculation.

5. Double or multiple stars modify the streak of the respective assiciated BSC star.

6. The color of the star modifies the variation within a streak, when crossing a filter boundary.

7. The timing of the exposures of subframes within one swath might differ from perfect periodicity.

8. Artifacts of lossy compression might modify the results. However, the considered image 117 doesn't show obvious artifacts.

9. Any deviation of the arrangement of the CCD pixels from a perfect square grid would modify orthonormality.

10. A decentered optics would require additional Brownian parameters.

4.2.3 Selected probe-intrinsic effects

1. Imperfections of the spacecraft (or camera) assembly might induce an additional camera rotation around the z-axis,

2. Nutation of the probe might induce a more complex motion than rotation around the x-axis.

4.2.4 Selected extrinsic effects

1. Inaccuracies in the BSC data should be cross-checked with other star catalogs. Stars with large errors, however, would have been discarded by the algorithm.

2. The proper motion of some stars may be sufficiently relevant to be considered.

3. The spacecraft moves relative to the BSC reference frame. This indices a shift in the star positions. Relevance should be checked.

4. Effects of General Relativity apear negligible.

5. Hits of ionized particles near a streak might shift blip centroids.

Related, but much more advanced methods have been applied e.g. to LROC [10] and GAIA [9], [1].

A Example Runs for Image JNCE_2014038_00R117_V01

The data of this section have been obtained by analysing image JNCE_2014038_00R117_V01. The red readout region is assumed to start at pixel line $y = 306$ of the CCD. The origin $(0, 0)$ of the coordinates is assumed to be at the lower left corner of an image, with positive y-axis upward, and with positive x-axis to the right. Blip positions and errors are measured in pixels, angles in radians. Rotations are described as 3x3-matrices, to be multiplied from the left to column vectors.

A.1 Set Optical Center Y to 600, and Include Brownian K1

A.1.1 Parameters

$$\begin{aligned}
\text{Rotation of frame 0:} \quad & \begin{pmatrix} 0.2502945906 & 0.8307565696 & 0.4861408121 \\ -0.5739091323 & -0.2862722959 & 0.7709770718 \\ 0.7797313163 & -0.4773800316 & 0.4114139833 \end{pmatrix} \\
p_4 = \quad & -0.077835572556 \\
p_5 = \quad & 1486.463013976616 \\
p_6 = \quad & 813.886967993660 \\
p_7 = \quad & 600.000000000000 \\
p_{8,2} = \quad & -3.118341654035 \cdot 10^{-8}
\end{aligned}$$

A.1.2 Samples

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4104	$(556.460, 205.279)$	$(562.688, 205.341)$	6.229	
HR 3923	$(882.870, 402.591)$	$(885.115, 397.290)$	5.757	
HR 3748	$(1179.098, 550.663)$	$(1199.891, 572.386)$	30.070	discarded
HR 4094	$(745.894, 563.085)$	$(770.126, 596.598)$	41.356	discarded
HR 4232	$(624.996, 662.217)$	$(632.226, 666.797)$	8.558	discarded
HR 4163	$(727.772, 689.147)$	$(739.931, 702.358)$	17.955	discarded
HR 4630	$(70.197, 732.108)$	$(86.766, 763.765)$	35.731	discarded
HR 3845	$(1217.762, 790.138)$	$(1224.016, 788.473)$	6.472	
HR 4382	$(474.316, 796.316)$	$(496.212, 830.344)$	40.464	discarded
HR 4662	$(90.801, 907.917)$	$(106.436, 934.804)$	31.103	discarded

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4368	$(631.174, 1070.538)$	$(649.225, 1097.728)$	32.636	discarded
HR 3950	$(1220.405, 1096.697)$	$(1221.679, 1090.722)$	6.110	
HR 3980	$(1200.621, 1179.470)$	$(1202.340, 1173.286)$	6.418	
HR 3982	$(1221.492, 1227.983)$	$(1223.166, 1222.395)$	5.833	
HR 4540	$(506.002, 1329.600)$	$(506.806, 1329.041)$	0.979	
HR 4386	$(730.694, 1328.930)$	$(731.541, 1326.574)$	2.504	
HR 3975	$(1286.949, 1353.604)$	$(1289.663, 1348.293)$	5.965	
HR 4517	$(598.319, 1432.787)$	$(598.703, 1432.377)$	0.562	
HR 4399	$(774.170, 1450.340)$	$(774.630, 1449.292)$	1.144	
HR 4057	$(1250.408, 1476.620)$	$(1252.057, 1472.178)$	4.739	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4359	$(891.039, 1530.870)$	$(891.131, 1529.744)$	1.131	
HR 4359	$(892.409, 1542.942)$	$(891.799, 1540.503)$	2.514	
HR 4608	$(520.913, 1561.600)$	$(520.633, 1561.579)$	0.280	
HR 4910	$(147.829, 1631.131)$	$(146.423, 1633.422)$	2.688	
HR 4534	$(691.950, 1635.468)$	$(692.123, 1635.927)$	0.490	
HR 4357	$(957.055, 1658.354)$	$(956.971, 1657.543)$	0.816	
HR 4357	$(958.587, 1669.447)$	$(958.209, 1668.354)$	1.156	
HR 4362	$(985.451, 1732.106)$	$(984.509, 1731.518)$	1.110	
HR 4932	$(239.198, 1854.648)$	$(237.513, 1857.107)$	2.981	
HR 4247	$(1235.324, 1962.931)$	$(1236.511, 1960.770)$	2.465	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4920	$(370.243, 1990.199)$	$(368.806, 1993.925)$	3.994	
HR 4377	$(1095.075, 1997.054)$	$(1094.410, 1996.938)$	0.675	
HR 4737	$(702.944, 2108.233)$	$(702.169, 2110.249)$	2.160	
HR 4069	$(1474.637, 2101.126)$	$(1477.592, 2098.720)$	3.811	
HR 5200	$(75.221, 2212.666)$	$(74.849, 2216.543)$	3.895	
HR 4954	$(511.196, 2254.877)$	$(510.783, 2259.009)$	4.153	
HR 4335	$(1273.602, 2276.218)$	$(1273.631, 2276.261)$	0.052	
HR 5235	$(110.791, 2289.045)$	$(109.951, 2293.813)$	4.841	
HR 5340	$(29.108, 2425.912)$	$(28.963, 2431.249)$	5.339	
HR 5340	$(21.876, 2439.942)$	$(22.619, 2444.487)$	4.605	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4915	(744.715, 2446.915)	(743.491, 2449.513)	2.872	
HR 4518	(1162.182, 2454.641)	(1161.310, 2455.756)	1.415	
HR 4846	(905.426, 2567.163)	(904.326, 2567.804)	1.274	
HR 4846	(933.838, 2577.539)	(904.326, 2567.804)	31.076	discarded
HR 4295	(1433.514, 2607.650)	(1434.847, 2607.408)	1.355	
HR 4554	(1211.158, 2621.933)	(1209.903, 2623.205)	1.787	
HR 5429	(234.817, 2735.147)	(234.411, 2738.955)	3.830	
HR 4660	(1186.737, 2752.989)	(1184.700, 2754.758)	2.697	
HR 5506	(107.761, 2756.184)	(107.718, 2760.350)	4.166	
HR 4905	(1054.004, 2813.113)	(1051.687, 2816.042)	3.735	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4905	(1055.953, 2824.876)	(1053.670, 2827.092)	3.182	
HR 5191	(782.899, 2854.208)	(781.273, 2857.897)	4.032	
HR 5435	(422.345, 2861.974)	(422.416, 2865.735)	3.761	
HR 5351	(640.233, 2907.317)	(639.454, 2910.648)	3.421	
HR 4434	(1464.957, 3021.500)	(1465.272, 3022.099)	0.677	
HR 5681	(184.924, 3027.876)	(185.570, 3031.673)	3.852	
HR 5404	(745.646, 3030.995)	(745.206, 3035.132)	4.160	
HR 5226	(1088.236, 3144.198)	(1085.299, 3146.343)	3.637	
HR 5291	(1057.681, 3167.047)	(1055.002, 3169.797)	3.839	
HR 5744	(817.914, 3345.209)	(816.610, 3347.205)	2.384	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 5589	(1018.743, 3337.820)	(1016.106, 3339.344)	3.046	
HR 5563	(1221.664, 3422.165)	(1219.098, 3423.335)	2.820	
HR 5735	(1139.350, 3450.269)	(1136.947, 3451.469)	2.686	
HR 5735	(1142.248, 3462.446)	(1139.701, 3462.629)	2.554	
HR 6132	(840.154, 3558.054)	(838.798, 3559.345)	1.873	
HR 6220	(234.795, 3594.414)	(228.219, 3606.753)	13.982	discarded
HR 6396	(945.528, 3702.271)	(943.662, 3703.150)	2.063	
HR 6418	(172.627, 3798.768)	(174.443, 3797.968)	1.985	
HR 6536	(605.706, 3821.283)	(606.270, 3821.448)	0.588	
HR 6688	(739.000, 3896.730)	(738.523, 3896.481)	0.538	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 6705	(605.000, 3941.226)	(605.723, 3940.826)	0.826	
HR 7001	(341.491, 4267.153)	(342.963, 4263.048)	4.362	
HR 7157	(509.251, 4290.751)	(510.684, 4287.473)	3.577	
HR 7139	(335.236, 4386.640)	(337.019, 4381.857)	5.105	
HR 7924	(835.000, 4697.929)	(834.770, 4693.244)	4.691	
HR 7417	(238.077, 4720.519)	(240.114, 4713.840)	6.984	
HR 7796	(678.000, 4726.622)	(679.002, 4721.395)	5.322	
HR 7796	(677.000, 4738.401)	(677.785, 4732.065)	6.384	

Number of included samples:	68
Number of discarded samples:	10
RMS error:	3.634230

A.2 Set Optical Center Y to 600, and Include Brownian K1 and K2

A.2.1 Parameters

$$\text{Rotation of frame 0:} \begin{pmatrix} 0.2500726651 & 0.8307401404 & 0.4862801217 \\ -0.5740510544 & -0.2864203703 & 0.7708175958 \\ 0.7796980499 & -0.4773197990 & 0.4115481469 \end{pmatrix}$$

$$p_4 = -0.077834135585$$
$$p_5 = 1492.565757374483$$
$$p_6 = 813.607125541612$$
$$p_7 = 600.000000000000$$
$$p_{8,2} = -6.040598817509 \cdot 10^{-8}$$
$$p_{8,4} = 3.967324089476 \cdot 10^{-14}$$

A.2.2 Samples

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4104	(556.460, 205.279)	(562.154, 205.274)	5.694	
HR 3923	(882.870, 402.591)	(885.193, 396.868)	6.176	
HR 3748	(1179.098, 550.663)	(1199.801, 572.492)	30.085	discarded
HR 4094	(745.894, 563.085)	(769.936, 596.227)	40.944	discarded
HR 4232	(624.996, 662.217)	(631.859, 666.583)	8.133	discarded
HR 4163	(727.772, 689.147)	(739.688, 701.983)	17.514	discarded
HR 4630	(70.197, 732.108)	(86.732, 764.058)	35.976	discarded
HR 3845	(1217.762, 790.138)	(1223.764, 788.708)	6.170	
HR 4382	(474.316, 796.316)	(495.988, 830.435)	40.419	discarded
HR 4662	(90.801, 907.917)	(106.427, 935.073)	31.331	discarded

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4368	(631.174, 1070.538)	(648.860, 1097.485)	32.232	discarded
HR 3950	(1220.405, 1096.697)	(1221.593, 1090.895)	5.922	
HR 3980	(1200.621, 1179.470)	(1202.251, 1173.497)	6.191	
HR 3982	(1221.492, 1227.983)	(1223.099, 1222.576)	5.640	
HR 4540	(506.002, 1329.600)	(506.656, 1329.104)	0.820	
HR 4386	(730.694, 1328.930)	(731.385, 1326.201)	2.815	
HR 3975	(1286.949, 1353.604)	(1289.273, 1348.602)	5.516	
HR 4517	(598.319, 1432.787)	(598.461, 1432.233)	0.573	
HR 4399	(774.170, 1450.340)	(774.600, 1448.885)	1.517	
HR 4057	(1250.408, 1476.620)	(1251.863, 1472.442)	4.424	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4359	(891.039, 1530.870)	(891.481, 1529.464)	1.474	
HR 4359	(892.409, 1542.942)	(892.062, 1540.123)	2.840	
HR 4608	(520.913, 1561.600)	(520.527, 1561.626)	0.386	
HR 4910	(147.829, 1631.131)	(147.201, 1633.800)	2.742	
HR 4534	(691.950, 1635.468)	(691.854, 1635.655)	0.211	
HR 4357	(957.055, 1658.354)	(957.500, 1657.314)	1.131	
HR 4357	(958.587, 1669.447)	(958.579, 1668.062)	1.385	
HR 4362	(985.451, 1732.106)	(985.012, 1731.256)	0.957	
HR 4932	(239.198, 1854.648)	(238.511, 1857.593)	3.024	
HR 4247	(1235.324, 1962.931)	(1236.375, 1961.083)	2.126	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4920	(370.243, 1990.199)	(369.258, 1994.249)	4.168	
HR 4377	(1095.075, 1997.054)	(1094.887, 1996.896)	0.246	
HR 4737	(702.944, 2108.233)	(701.982, 2109.919)	1.942	
HR 4069	(1474.637, 2101.126)	(1477.231, 2099.069)	3.310	
HR 5200	(75.221, 2212.666)	(74.309, 2216.510)	3.951	
HR 4954	(511.196, 2254.877)	(510.591, 2259.016)	4.183	
HR 4335	(1273.602, 2276.218)	(1273.420, 2276.581)	0.406	
HR 5235	(110.791, 2289.045)	(110.414, 2294.039)	5.009	
HR 5340	(29.108, 2425.912)	(27.615, 2431.153)	5.450	
HR 5340	(21.876, 2439.942)	(20.079, 2443.803)	4.259	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4915	(744.715, 2446.915)	(743.432, 2449.135)	2.565	
HR 4518	(1162.182, 2454.641)	(1161.457, 2455.955)	1.500	
HR 4846	(905.426, 2567.163)	(904.630, 2567.465)	0.852	
HR 4846	(933.838, 2577.539)	(904.630, 2567.465)	30.896	discarded
HR 4295	(1433.514, 2607.650)	(1434.205, 2607.915)	0.740	
HR 4554	(1211.158, 2621.933)	(1209.929, 2623.479)	1.975	
HR 5429	(234.817, 2735.147)	(235.387, 2739.396)	4.287	
HR 4660	(1186.737, 2752.989)	(1184.837, 2754.978)	2.750	
HR 5506	(107.761, 2756.184)	(107.967, 2760.504)	4.325	
HR 4905	(1054.004, 2813.113)	(1052.280, 2816.000)	3.363	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4905	$(1055.953, 2824.876)$	$(1054.030, 2827.048)$	2.901	
HR 5191	$(782.899, 2854.208)$	$(781.261, 2857.503)$	3.680	
HR 5435	$(422.345, 2861.974)$	$(422.608, 2865.943)$	3.977	
HR 5351	$(640.233, 2907.317)$	$(639.086, 2910.423)$	3.312	
HR 4434	$(1464.957, 3021.500)$	$(1464.680, 3022.579)$	1.114	
HR 5681	$(184.924, 3027.876)$	$(186.491, 3032.007)$	4.418	
HR 5404	$(745.646, 3030.995)$	$(745.045, 3034.807)$	3.859	
HR 5226	$(1088.236, 3144.198)$	$(1085.718, 3146.349)$	3.312	
HR 5291	$(1057.681, 3167.047)$	$(1055.502, 3169.747)$	3.469	
HR 5744	$(817.914, 3345.209)$	$(816.633, 3346.817)$	2.056	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 5589	$(1018.743, 3337.820)$	$(1016.427, 3339.231)$	2.712	
HR 5563	$(1221.664, 3422.165)$	$(1219.045, 3423.648)$	3.010	
HR 5735	$(1139.350, 3450.269)$	$(1137.285, 3451.607)$	2.461	
HR 5735	$(1142.248, 3462.446)$	$(1139.762, 3462.863)$	2.521	
HR 6132	$(840.154, 3558.054)$	$(838.861, 3559.049)$	1.632	
HR 6220	$(234.795, 3594.414)$	$(229.046, 3607.174)$	13.995	discarded
HR 6396	$(945.528, 3702.271)$	$(944.014, 3702.977)$	1.670	
HR 6418	$(172.627, 3798.768)$	$(175.176, 3798.245)$	2.603	
HR 6536	$(605.706, 3821.283)$	$(605.714, 3821.280)$	0.008	
HR 6688	$(739.000, 3896.730)$	$(738.223, 3896.125)$	0.985	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 6705	$(605.000, 3941.226)$	$(605.155, 3940.649)$	0.597	
HR 7001	$(341.491, 4267.153)$	$(343.310, 4263.386)$	4.183	
HR 7157	$(509.251, 4290.751)$	$(510.242, 4287.451)$	3.445	
HR 7139	$(335.236, 4386.640)$	$(337.380, 4382.221)$	4.912	
HR 7924	$(835.000, 4697.929)$	$(834.568, 4692.932)$	5.015	
HR 7417	$(238.077, 4720.519)$	$(240.675, 4714.137)$	6.891	
HR 7796	$(678.000, 4726.622)$	$(678.319, 4721.168)$	5.463	
HR 7796	$(677.000, 4738.401)$	$(677.260, 4731.795)$	6.611	

Number of included samples:	68
Number of discarded samples:	10
RMS error:	3.582638

A.3 Include Optical Center Y, and Brownian K1

A.3.1 Parameters

$$\text{Rotation of frame 0:} \begin{pmatrix} 0.2506242196 & 0.8307643699 & 0.4859616074 \\ -0.6347046579 & -0.2466465099 & 0.7355383620 \\ 0.7309839244 & -0.4989950509 & 0.4720430427 \end{pmatrix}$$

$$\begin{aligned} p_4 &= -0.077835602103 \\ p_5 &= 1498.589175567390 \\ p_6 &= 814.276331289716 \\ p_7 &= 480.496273868451 \\ p_{8,2} &= -3.687397612268 \cdot 10^{-8} \end{aligned}$$

A.3.2 Samples

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4104	$(556.460, 205.279)$	$(562.252, 205.454)$	5.795	
HR 3923	$(882.870, 402.591)$	$(885.168, 398.014)$	5.122	
HR 3748	$(1179.098, 550.663)$	$(1200.269, 572.459)$	30.385	discarded
HR 4094	$(745.894, 563.085)$	$(770.023, 596.979)$	41.605	discarded
HR 4232	$(624.996, 662.217)$	$(632.220, 667.267)$	8.814	discarded
HR 4163	$(727.772, 689.147)$	$(739.789, 702.771)$	18.167	discarded
HR 4630	$(70.197, 732.108)$	$(85.680, 761.979)$	33.645	discarded
HR 3845	$(1217.762, 790.138)$	$(1223.540, 788.719)$	5.949	
HR 4382	$(474.316, 796.316)$	$(495.745, 830.380)$	40.243	discarded
HR 4662	$(90.801, 907.917)$	$(107.941, 933.559)$	30.843	discarded

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4368	$(631.174, 1070.538)$	$(648.754, 1098.023)$	32.626	discarded
HR 3950	$(1220.405, 1096.697)$	$(1222.028, 1090.680)$	6.232	
HR 3980	$(1200.621, 1179.470)$	$(1201.863, 1173.548)$	6.051	
HR 3982	$(1221.492, 1227.983)$	$(1223.560, 1222.326)$	6.023	
HR 4540	$(506.002, 1329.600)$	$(506.439, 1329.190)$	0.599	
HR 4386	$(730.694, 1328.930)$	$(731.322, 1327.053)$	1.979	
HR 3975	$(1286.949, 1353.604)$	$(1289.890, 1348.028)$	6.304	
HR 4517	$(598.319, 1432.787)$	$(598.586, 1432.844)$	0.274	
HR 4399	$(774.170, 1450.340)$	$(774.454, 1449.822)$	0.591	
HR 4057	$(1250.408, 1476.620)$	$(1252.254, 1472.031)$	4.946	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4359	$(891.039, 1530.870)$	$(891.346, 1530.141)$	0.792	
HR 4359	$(892.409, 1542.942)$	$(891.584, 1541.280)$	1.855	
HR 4608	$(520.913, 1561.600)$	$(520.558, 1561.911)$	0.472	
HR 4910	$(147.829, 1631.131)$	$(145.690, 1632.142)$	2.366	
HR 4534	$(691.950, 1635.468)$	$(691.521, 1636.253)$	0.895	
HR 4357	$(957.055, 1658.354)$	$(957.467, 1657.889)$	0.621	
HR 4357	$(958.587, 1669.447)$	$(957.910, 1669.091)$	0.764	
HR 4362	$(985.451, 1732.106)$	$(984.682, 1731.867)$	0.805	
HR 4932	$(239.198, 1854.648)$	$(237.281, 1856.400)$	2.597	
HR 4247	$(1235.324, 1962.931)$	$(1236.235, 1960.741)$	2.371	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4920	$(370.243, 1990.199)$	$(368.033, 1993.664)$	4.110	
HR 4377	$(1095.075, 1997.054)$	$(1094.825, 1997.083)$	0.251	
HR 4737	$(702.944, 2108.233)$	$(701.786, 2110.661)$	2.691	
HR 4069	$(1474.637, 2101.126)$	$(1476.434, 2097.736)$	3.837	
HR 5200	$(75.221, 2212.666)$	$(76.450, 2215.324)$	2.929	
HR 4954	$(511.196, 2254.877)$	$(509.885, 2259.110)$	4.432	
HR 4335	$(1273.602, 2276.218)$	$(1274.532, 2275.866)$	0.994	
HR 5235	$(110.791, 2289.045)$	$(108.757, 2292.397)$	3.922	
HR 5340	$(29.108, 2425.912)$	$(27.972, 2429.372)$	3.642	
HR 5340	$(21.876, 2439.942)$	$(25.793, 2443.323)$	5.175	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4915	$(744.715, 2446.915)$	$(743.349, 2450.190)$	3.549	
HR 4518	$(1162.182, 2454.641)$	$(1160.921, 2455.987)$	1.844	
HR 4846	$(905.426, 2567.163)$	$(904.116, 2568.528)$	1.892	
HR 4846	$(933.838, 2577.539)$	$(904.116, 2568.528)$	31.057	discarded
HR 4295	$(1433.514, 2607.650)$	$(1433.921, 2606.563)$	1.161	
HR 4554	$(1211.158, 2621.933)$	$(1210.143, 2623.074)$	1.527	
HR 5429	$(234.817, 2735.147)$	$(234.627, 2738.430)$	3.289	
HR 4660	$(1186.737, 2752.989)$	$(1185.043, 2754.671)$	2.387	
HR 5506	$(107.761, 2756.184)$	$(108.053, 2759.157)$	2.987	
HR 4905	$(1054.004, 2813.113)$	$(1052.682, 2816.214)$	3.371	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4905	$(1055.953, 2824.876)$	$(1053.352, 2827.608)$	3.772	
HR 5191	$(782.899, 2854.208)$	$(781.137, 2858.435)$	4.580	
HR 5435	$(422.345, 2861.974)$	$(422.128, 2865.805)$	3.837	
HR 5351	$(640.233, 2907.317)$	$(638.783, 2910.965)$	3.926	
HR 4434	$(1464.957, 3021.500)$	$(1465.136, 3020.887)$	0.638	
HR 5681	$(184.924, 3027.876)$	$(185.042, 3030.806)$	2.932	
HR 5404	$(745.646, 3030.995)$	$(744.910, 3035.523)$	4.588	
HR 5226	$(1088.236, 3144.198)$	$(1085.799, 3146.446)$	3.315	
HR 5291	$(1057.681, 3167.047)$	$(1055.736, 3169.922)$	3.471	
HR 5744	$(817.914, 3345.209)$	$(816.605, 3347.877)$	2.972	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 5589	$(1018.743, 3337.820)$	$(1015.949, 3339.896)$	3.481	
HR 5563	$(1221.664, 3422.165)$	$(1220.100, 3423.028)$	1.786	
HR 5735	$(1139.350, 3450.269)$	$(1138.306, 3451.402)$	1.541	
HR 5735	$(1142.248, 3462.446)$	$(1139.280, 3462.983)$	3.017	
HR 6132	$(840.154, 3558.054)$	$(838.949, 3559.725)$	2.061	
HR 6220	$(234.795, 3594.414)$	$(229.442, 3606.503)$	13.221	discarded
HR 6396	$(945.528, 3702.271)$	$(944.341, 3703.452)$	1.674	
HR 6418	$(172.627, 3798.768)$	$(174.236, 3797.131)$	2.295	
HR 6536	$(605.706, 3821.283)$	$(605.587, 3821.755)$	0.486	
HR 6688	$(739.000, 3896.730)$	$(738.551, 3896.960)$	0.505	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 6705	$(605.000, 3941.226)$	$(605.160, 3941.133)$	0.185	
HR 7001	$(341.491, 4267.153)$	$(343.468, 4263.057)$	4.549	
HR 7157	$(509.251, 4290.751)$	$(510.525, 4287.725)$	3.283	
HR 7139	$(335.236, 4386.640)$	$(337.791, 4381.917)$	5.371	
HR 7924	$(835.000, 4697.929)$	$(835.250, 4693.633)$	4.303	
HR 7417	$(238.077, 4720.519)$	$(239.453, 4713.245)$	7.403	
HR 7796	$(678.000, 4726.622)$	$(678.859, 4721.776)$	4.921	
HR 7796	$(677.000, 4738.401)$	$(678.392, 4732.921)$	5.654	

Number of included samples:	68
Number of discarded samples:	10
RMS error:	3.516436

A.4 Include Optical Center Y, and Brownian K1 and K2

A.4.1 Parameters

$$\text{Rotation of frame 0:} \begin{pmatrix} 0.2504096126 & 0.8307700647 & 0.4860601669 \\ -0.6312882873 & -0.2491778962 & 0.7376647354 \\ 0.7340096214 & -0.4977263060 & 0.4686109818 \end{pmatrix}$$

$$\begin{aligned} p_4 &= -0.077835323987 \\ p_5 &= 1504.775501268894 \\ p_6 &= 814.039010579562 \\ p_7 &= 487.662207828404 \\ p_{8,2} &= -7.157963248833 \cdot 10^{-8} \\ p_{8,4} &= 5.017409363776 \cdot 10^{-14} \end{aligned}$$

A.4.2 Samples

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4104	(556.460, 205.279)	(561.670, 205.457)	5.214	
HR 3923	(882.870, 402.591)	(885.412, 397.603)	5.598	
HR 3748	(1179.098, 550.663)	(1200.286, 572.508)	30.432	discarded
HR 4094	(745.894, 563.085)	(769.855, 596.827)	41.384	discarded
HR 4232	(624.996, 662.217)	(631.721, 667.050)	8.281	discarded
HR 4163	(727.772, 689.147)	(739.527, 702.548)	17.826	discarded
HR 4630	(70.197, 732.108)	(85.651, 762.302)	33.920	discarded
HR 3845	(1217.762, 790.138)	(1223.430, 788.831)	5.816	
HR 4382	(474.316, 796.316)	(495.459, 830.442)	40.144	discarded
HR 4662	(90.801, 907.917)	(108.118, 933.873)	31.202	discarded

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4368	(631.174, 1070.538)	(648.302, 1097.905)	32.285	discarded
HR 3950	(1220.405, 1096.697)	(1221.982, 1090.781)	6.123	
HR 3980	(1200.621, 1179.470)	(1201.928, 1173.639)	5.975	
HR 3982	(1221.492, 1227.983)	(1223.517, 1222.434)	5.907	
HR 4540	(506.002, 1329.600)	(506.171, 1329.203)	0.431	
HR 4386	(730.694, 1328.930)	(731.103, 1326.776)	2.192	
HR 3975	(1286.949, 1353.604)	(1289.417, 1348.222)	5.921	
HR 4517	(598.319, 1432.787)	(598.163, 1432.660)	0.201	
HR 4399	(774.170, 1450.340)	(774.401, 1449.504)	0.867	
HR 4057	(1250.408, 1476.620)	(1252.041, 1472.194)	4.718	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4359	(891.039, 1530.870)	(891.760, 1530.135)	1.030	
HR 4359	(892.409, 1542.942)	(891.976, 1540.863)	2.124	
HR 4608	(520.913, 1561.600)	(520.276, 1561.868)	0.691	
HR 4910	(147.829, 1631.131)	(146.702, 1632.422)	1.713	
HR 4534	(691.950, 1635.468)	(691.189, 1636.185)	1.047	
HR 4357	(957.055, 1658.354)	(958.077, 1657.896)	1.120	
HR 4357	(958.587, 1669.447)	(958.481, 1668.742)	0.712	
HR 4362	(985.451, 1732.106)	(985.338, 1731.716)	0.406	
HR 4932	(239.198, 1854.648)	(238.527, 1856.717)	2.175	
HR 4247	(1235.324, 1962.931)	(1236.136, 1960.930)	2.159	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4920	(370.243, 1990.199)	(368.565, 1993.848)	4.017	
HR 4377	(1095.075, 1997.054)	(1095.421, 1997.078)	0.347	
HR 4737	(702.944, 2108.233)	(701.489, 2110.449)	2.651	
HR 4069	(1474.637, 2101.126)	(1475.826, 2098.063)	3.286	
HR 5200	(75.221, 2212.666)	(75.915, 2215.421)	2.840	
HR 4954	(511.196, 2254.877)	(509.603, 2259.120)	4.532	
HR 4335	(1273.602, 2276.218)	(1274.157, 2276.072)	0.573	
HR 5235	(110.791, 2289.045)	(109.293, 2292.583)	3.843	
HR 5340	(29.108, 2425.912)	(26.026, 2429.516)	4.742	
HR 5340	(21.876, 2439.942)	(23.022, 2443.028)	3.292	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4915	(744.715, 2446.915)	(743.183, 2449.805)	3.271	
HR 4518	(1162.182, 2454.641)	(1161.219, 2456.067)	1.720	
HR 4846	(905.426, 2567.163)	(904.529, 2568.145)	1.330	
HR 4846	(933.838, 2577.539)	(904.529, 2568.145)	30.777	discarded
HR 4295	(1433.514, 2607.650)	(1433.006, 2606.971)	0.848	
HR 4554	(1211.158, 2621.933)	(1210.179, 2623.250)	1.641	
HR 5429	(234.817, 2735.147)	(235.806, 2738.704)	3.692	
HR 4660	(1186.737, 2752.989)	(1185.213, 2754.816)	2.379	
HR 5506	(107.761, 2756.184)	(108.388, 2759.311)	3.189	
HR 4905	(1054.004, 2813.113)	(1053.295, 2816.306)	3.271	

BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 4905	$(1055.953, 2824.876)$	$(1053.921, 2827.466)$	3.292	
HR 5191	$(782.899, 2854.208)$	$(781.064, 2858.116)$	4.317	
HR 5435	$(422.345, 2861.974)$	$(422.241, 2865.886)$	3.913	
HR 5351	$(640.233, 2907.317)$	$(638.290, 2910.882)$	4.061	
HR 4434	$(1464.957, 3021.500)$	$(1464.259, 3021.276)$	0.733	
HR 5681	$(184.924, 3027.876)$	$(186.209, 3030.995)$	3.373	
HR 5404	$(745.646, 3030.995)$	$(744.669, 3035.393)$	4.505	
HR 5226	$(1088.236, 3144.198)$	$(1086.317, 3146.470)$	2.974	
HR 5291	$(1057.681, 3167.047)$	$(1056.305, 3169.959)$	3.221	
HR 5744	$(817.914, 3345.209)$	$(816.604, 3347.484)$	2.625	
BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 5589	$(1018.743, 3337.820)$	$(1016.462, 3339.696)$	2.953	
HR 5563	$(1221.664, 3422.165)$	$(1219.956, 3423.246)$	2.021	
HR 5735	$(1139.350, 3450.269)$	$(1138.595, 3451.556)$	1.492	
HR 5735	$(1142.248, 3462.446)$	$(1139.522, 3463.051)$	2.792	
HR 6132	$(840.154, 3558.054)$	$(839.002, 3559.654)$	1.972	
HR 6220	$(234.795, 3594.414)$	$(230.437, 3606.773)$	13.105	discarded
HR 6396	$(945.528, 3702.271)$	$(944.730, 3703.486)$	1.453	
HR 6418	$(172.627, 3798.768)$	$(175.202, 3797.280)$	2.975	
HR 6536	$(605.706, 3821.283)$	$(604.922, 3821.716)$	0.895	
HR 6688	$(739.000, 3896.730)$	$(738.143, 3896.699)$	0.858	
BSC Star	Assigned Blip Pos	Simulated Blip Pos	Error	Note
HR 6705	$(605.000, 3941.226)$	$(604.470, 3941.070)$	0.553	
HR 7001	$(341.491, 4267.153)$	$(343.835, 4263.215)$	4.584	
HR 7157	$(509.251, 4290.751)$	$(509.934, 4287.662)$	3.164	
HR 7139	$(335.236, 4386.640)$	$(338.168, 4382.093)$	5.411	
HR 7924	$(835.000, 4697.929)$	$(835.054, 4693.491)$	4.438	
HR 7417	$(238.077, 4720.519)$	$(240.305, 4713.359)$	7.499	
HR 7796	$(678.000, 4726.622)$	$(678.090, 4721.735)$	4.888	
HR 7796	$(677.000, 4738.401)$	$(677.670, 4732.526)$	5.913	

Number of included samples:	68
Number of discarded samples:	10
RMS error:	3.449727

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