Some Empirical Approximations of the Stray Light in Junocam Image EFB03, Part II

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Abstract

The Junocam image EFB03 shows mostly stray light. This stray light can be approximated by sets of 1-dimensional linear functions, the parameters of which can be roughly approximated by 1-dimensional Gauss functions. Junocam rotates while taking images. To account for this, it appears appropriate to work with an adjustment of the Gauss functions to this periodicity. It turns out, however, that the induced effect is small.

The parameters of the 1-dimensional linear functions show skewness and kurtosis. To account for kurtosis, a family of power-law functions is investigated. To account for the skewness of the symmetrical power-law functions the parameter can first be transformed e.g. by an appropriate polyline, or by a Fourier series. Determining the according parameters appears to require numerical methods. Providing partial derivatives of the power-law functions prepares an application of quasi-Newton methods.

A family of generalized Gauss function as another approach to account for skewness and kurtosis is mentioned. [1]

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[1]This document was typeset with LaTeX.

This article has been published as part of a post at `unmannedspaceflight.com`.

Contents

1 Introduction

Images taken by Juno's Education and Outreach camera Junocam during the Earth flyby (EFB) in October 2013 are providing a first publicly available set of in-flight tests, similar to the images expected to be taken during Juno's Jupiter mission starting in mid-2016. So this article may be put into the context of [1, subsection 6.4], goal 3: "Provide data to the amateur image processing community and encourage them to produce a variety of products". Raw images are provided online by Malin Space Science Systems, San Diego, CA, USA (MSSS) [2].

This article is the second part of attempts to approximate the stray light in EFB03 by empirical functions. Section 2 replaces the Gauss functions of part I by a periodic version of Gauss functions, to account for the rotation of Junocam.

Section 3 investigates a family of functions which accounts for skewness and kurtosis, since a closer look to the stray light of EFB03 reveals deviations from Gauss functions in both these aspects. This approach is considerably more complex than using a Gauss function, and it appears to require numerical methods to be solved. The section provides the according methods and calculations. The reader will need some experience with calculus, Fourier series, and the n-dimensional Newton method.

Section 4 provides some initial steps to an investigation of a family of generalized Gauss functions, as alternative approach to account for skewness and kurtosis. Methods for further elaboration have been provided in section 3. At this point it's not yet obvious, whether such an elaboration will become useful or necessary, since the application of the power-law family of function looks like being more suitable to approximate the stray light than the considered family of generalized Gauss functions; this is conjectural, however.

The appendices list some experimental results.

2 Periodic Version of Gauss Functions

2.1 Definition and Basic Properties

The Gauss function

$$G_{a,\mu,\sigma} : \mathbf{R} \rightarrow \mathbf{R}^+, \tag{1}$$

$$x \mapsto a \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \tag{2}$$

with $a, \mu \in \mathbf{R}$, and $\sigma \in \mathbf{R}^+$, shows one local maximum (peak) at μ.

For $x \in \mathbf{R}$, $x > \mu$,

$$\sigma = \frac{x-\mu}{\sqrt{2 \ln \frac{a}{G_{a,\mu,\sigma}(x)}}}. \tag{3}$$

Let $r \in \mathbf{R}^+$. A function $\tilde{G}$ related to the Gauss function, but with the property

$$\tilde{G}(x) = \tilde{G}(x + 2\pi r), \tag{4}$$

for any $x \in \mathbf{R}$, can be defined by

$$\tilde{G}_{a,\tilde{\mu},\tilde{\sigma}} : \mathbf{R} \rightarrow \mathbf{R}^+, \tag{5}$$

$$x \mapsto a \cdot e^{-\frac{1}{2}\left(\frac{2r}{\tilde{\sigma}} \sin \frac{x-\tilde{\mu}}{2r}\right)^2}, \tag{6}$$

with $a, \tilde{\mu} \in \mathbf{R}$, and $\tilde{\sigma} \in \mathbf{R}^+$. The periodicity $\tilde{G}$ is induced by the π-periodicity of the squared sinus function. The similarity to the Gauss functions is immediate for $\frac{x-\tilde{\mu}}{2r} \rightarrow 0$, since $\sin x \rightarrow x$, for $x \rightarrow 0$.

For $x \in [\tilde{\mu}, \tilde{\mu} + \pi r)$,

$$\tilde{\sigma} = \frac{2r \sin \frac{x-\tilde{\mu}}{2r}}{\sqrt{2 \ln \frac{a}{\tilde{G}_{a,\tilde{\mu},\tilde{\sigma}}(x)}}}. \tag{7}$$

2.2 Application to EFB03

Replacing the Gauss functions for the approximations of EFB03 of part I of this article by the periodic version results in perceivable, but small relative changes of the parameters, roughly 1% for sigma. This is mainly due to the half number 79/2 of framelets per camera rotation being about $\frac{79}{2} \cdot \frac{1}{4.6} \approx 8.6$, with a sigma of up to about 4.6 framelets. I.e. one camera rotation is about ± 8.6 sigma of the approximating Gauss functions. Appendix A lists some of the calculated parameters.

3 A Family of 1-Dimensional Power-Law Peaks

3.1 Motivation

Closer comparison of the EFB03 stray light with Gauss functions reveals two properties of the stray light which are difficult to express with modified Gauss functions. These two properties can roughly subsumed as skewness and kurtosis. Skewness means an asymmetry with respect to the maximum of the function. Kurtosis means shoulders, in this case the stray light functions are wider at their basis than Gauss functions of otherwise similar shape. This could be accounted for, e.g. by a sum of several Gauss functions. Each Gauss function is described by essentially three parameters. The degrees of freedom of this family of functions are hence 3-fold the number of summed Gauss functions.

This section investigates an alternative approach with another family of functions, intended to obtain good approximations with a reduced number of degrees of freedom relative to equivalent descriptions by sums of Gauss functions.

The starting point is inspired by stray light interpreted as the result of a point light source emitting radialsymmetric light from some distance to the CCD plane. With d the distance of the point light source from the CCD, and x the distance on the CCD to the point on the CCD next to the point light source, a formula similar to

$$h(x) = \frac{1}{\sqrt{(\frac{x}{d})^2 + 1}} \tag{8}$$

in the simplest case, can be derived geometrically. Experiments with this type of formulas showed a kurtosis much worse than the Gauss functions. The desired kurtosis was between the Gauss functions and the newly investigated type of functions, however much closer to the Gauss function type.

Further experiments with generalized versions of equation (8) showed, that it's possible to heal this deficite, with results significantly better than according Gauss functions. Equation (8) can be re-written as

$$h_{a,\alpha,\beta,\mu,\varrho}(x) = a \cdot \left(\left(\frac{\mid x - \mu \mid}{\varrho} \right)^{\alpha} + 1 \right)^{-\beta}, \tag{9}$$

with $a = 1$, $\alpha = 2$, $\beta = \frac{1}{2}$, $\mu = 0$, and $\varrho = d$. Adjusting these five parameters appropriately appeared to allow for good approximations of the 1-dimensional stray light functions in terms of kurtosis.

Since functions of type (9) are symmetric with μ as axis, they don't allow adjusting for skewness. Another generalization step appeared recommended. This time inspired by the periodicity of 1-dimensional stray light functions, as discussed in section 2, and by the wish to add asymmetry. Both properties, periodicity, and asymmetry, can be modeled with Fourier series. One approach would be a description of the whole stray light function as a Fourier series. But this would make interpretations harder, and it would be less obvious, how an appropriate family of Fourier series should be chosen which allows for good approximations and for a low number of degrees of freedom at the same time.

Just replacing the $| x - \mu |$ part of equation (9) appeared intuitively easier to access, and it generalizes the underlying idea of section 2, which has already been shown to be applicable.

As a very rough approach to account for skewness may serve a replacement of $| x - \mu |$ by $u_1 \cdot | x - \mu |$, for $x \le \mu$, and $u_2 \cdot | x - \mu |$, for $x \ge \mu$, with $u_1, u_2 > 0$. For $x = \mu$, $u_1 \cdot | x - \mu | = 0 = u_2 \cdot | x - \mu |$.

This approach will first be translated into a periodic version by Fourier analysis. The resulting Fourier series will then be smoothed out, to remove the edge at $x = \mu$. The result will be a family of Fourier series with three parameters, u_1, u_2, and a parameter ζ for the smoothness at $x = \mu$. A 1-dimensional stray light curve (for a fixed vertical offset within each framelet) will then be approximated by a total of eight parameters, with one parameter probably redundant, since coupled to ϱ. So, choose e.g. $u_2 := \frac{1}{u_1}$, to reduce the number of parameters for the 1-dimensional stray light curve to seven. That's still a data reduction ratio of $79 : 7 \approx 11 : 1$. This choice forces either $u_1 \le 1 \le u_2$, or $u_2 \le 1 \le u_1$. Smoothed versions of the Fourier series can be constrained to those with derivative 1 at $x = \mu$. If necessary, this approach can be generalized further.

An obvious way to determine the parameters algebraically doesn't seem to be readily available. Therefore an rms minization method will be applied.

3.2 Some Similarity to Gauss Functions

Gauss functions can be approximated in the following sense by some functions h as defined in equation (9):

Lemma 1 *Let*

$$\alpha_0 := 4 \cdot \ln 2, \tag{10}$$

and

$$\beta_0 := \frac{1}{2 \cdot \ln 2}. \tag{11}$$

Let $\sigma > 0$. Then, for $x = \mu$, and for $x = \mu \pm \sigma$:

$$h_{a,\alpha_0,\beta_0,\mu,\sigma}(x) = G_{a,\mu,\sigma}(x), \tag{12}$$

and

$$h'_{a,\alpha_0,\beta_0,\mu,\sigma}(x) = G'_{a,\mu,\sigma}(x). \tag{13}$$

Straightforward calculation show the lemma:

$$h_{a,\alpha_0,\beta_0,\mu,\sigma}(\mu) = G_{a,\mu,\sigma}(\mu), \tag{14}$$

since

$$
\begin{aligned}
h_{a,\alpha_0,\beta_0,\mu,\sigma}(\mu) &= a \cdot \left(\left(\frac{\mid \mu - \mu \mid}{\sigma}\right)^{\alpha_0} + 1\right)^{-\beta_0} && (15)\\
&= a \cdot (0^{\alpha_0} + 1)^{-\beta_0} && (16)\\
&= a \cdot 1 && (17)\\
&= a, && (18)
\end{aligned}
$$

and

$$
\begin{aligned}
G_{a,\mu,\sigma}(\mu) &= a \cdot e^{-\frac{1}{2}\left(\frac{\mu-\mu}{\sigma}\right)^2} && (19)\\
&= a \cdot e^{-\frac{1}{2}\cdot 0} && (20)\\
&= a \cdot 1 && (21)\\
&= a. && (22)
\end{aligned}
$$

$$
h_{a,\alpha_0,\beta_0,\mu,\sigma}(\mu \pm \sigma) = G_{a,\mu,\sigma}(\mu \pm \sigma), \qquad (23)
$$

since

$$
\begin{aligned}
h_{a,\alpha_0,\beta_0,\mu,\sigma}(\mu \pm \sigma) &= a \cdot \left(\left(\frac{\mid \mu \pm \sigma - \mu \mid}{\sigma}\right)^{\alpha_0} + 1\right)^{-\beta_0} && (24)\\
&= a \cdot (1^{\alpha_0} + 1)^{-\beta_0} && (25)\\
&= a \cdot 2^{-\beta_0} && (26)\\
&= a \cdot 2^{-\frac{1}{2\cdot\ln 2}} && (27)\\
&= a \cdot e^{-\frac{ln2}{2\cdot\ln 2}} && (28)\\
&= a \cdot e^{-\frac{1}{2}}, && (29)
\end{aligned}
$$

and

$$
\begin{aligned}
G_{a,\mu,\sigma}(\mu \pm \sigma) &= a \cdot e^{-\frac{1}{2}\left(\frac{\mu\pm\sigma-\mu}{\sigma}\right)^2} && (30)\\
&= a \cdot e^{-\frac{1}{2}\cdot 1^2} && (31)\\
&= a \cdot e^{-\frac{1}{2}}. && (32)
\end{aligned}
$$

For $x \geq \mu$, and by the chain rule,

$$
\begin{aligned}
h'_{a,\alpha,\beta,\mu,\sigma}(x) &= \left(a \cdot \left(\left(\frac{\mid x - \mu \mid}{\sigma}\right)^{\alpha} + 1\right)^{-\beta}\right)' && (33)\\
&= \left(a \cdot \left(\left(\frac{x - \mu}{\sigma}\right)^{\alpha} + 1\right)^{-\beta}\right)' && (34)\\
&= a \cdot (-\beta)\left(\left(\frac{x - \mu}{\sigma}\right)^{\alpha} + 1\right)^{-\beta-1} \cdot \alpha \cdot \left(\frac{x - \mu}{\sigma}\right)^{\alpha-1} \cdot \frac{1}{\sigma} && (35)\\
&= -\frac{a\alpha\beta}{\sigma} \cdot \left(\left(\frac{x - \mu}{\sigma}\right)^{\alpha} + 1\right)^{-\beta-1} \cdot \left(\frac{x - \mu}{\sigma}\right)^{\alpha-1}, && (36)
\end{aligned}
$$

and

$$G'_{a,\mu,\sigma}(x) = \left(a \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right)' \tag{37}$$
$$= a \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot \left(-\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma} \tag{38}$$
$$= -a \cdot \frac{x-\mu}{\sigma^2} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}. \tag{39}$$

$$h'_{a,\alpha_0,\beta_0,\mu,\sigma}(\mu) = G'_{a,\mu,\sigma}(\mu), \tag{40}$$

since

$$h'_{a,\alpha,\beta,\mu,\sigma}(\mu) = -\frac{a\alpha\beta}{\sigma} \cdot \left(\left(\frac{\mu-\mu}{\sigma}\right)^{\alpha} + 1\right)^{-\beta-1} \cdot \left(\frac{\mu-\mu}{\sigma}\right)^{\alpha-1} \tag{41}$$
$$= -\frac{a\alpha\beta}{\sigma} \cdot \left(\left(\frac{\mu-\mu}{\sigma}\right)^{\alpha} + 1\right)^{-\beta-1} \cdot 0 \tag{42}$$
$$= 0, \tag{43}$$

for any α, β not causing singularities, and

$$G'_{a,\mu,\sigma}(\mu) = -a \cdot \frac{\mu-\mu}{\sigma^2} \cdot e^{-\frac{1}{2}\left(\frac{\mu-\mu}{\sigma}\right)^2} \tag{44}$$
$$= -a \cdot 0 \cdot e^{-\frac{1}{2}\left(\frac{\mu-\mu}{\sigma}\right)^2} \tag{45}$$
$$= 0. \tag{46}$$

$$h'_{a,\alpha_0,\beta_0,\mu,\sigma}(\mu+\sigma) = G'_{a,\mu,\sigma}(\mu+\sigma), \tag{47}$$

since

$$h'_{a,\alpha_0,\beta_0,\mu,\sigma}(\mu+\sigma) = -\frac{a\alpha_0\beta_0}{\sigma} \cdot \left(\left(\frac{\mu+\sigma-\mu}{\sigma}\right)^{\alpha_0} + 1\right)^{-\beta_0-1} \cdot \left(\frac{\mu+\sigma-\mu}{\sigma}\right)^{\alpha_0-1} \tag{48}$$
$$= -\frac{a\alpha_0\beta_0}{\sigma} \cdot (1^{\alpha_0}+1)^{-\beta_0-1} \cdot 1^{\alpha_0-1} \tag{49}$$
$$= -\frac{a\alpha_0\beta_0}{\sigma} \cdot 2^{-\beta_0-1} \tag{50}$$
$$= -a \cdot \frac{4 \cdot \ln 2 \cdot \frac{1}{2 \cdot \ln 2}}{\sigma} \cdot 2^{-\frac{1}{2\cdot\ln 2}-1} \tag{51}$$
$$= -a \cdot \frac{2}{\sigma} \cdot 2^{-\frac{1}{2\cdot\ln 2}-1} \tag{52}$$
$$= -\frac{a}{\sigma} \cdot 2^{-\frac{1}{2\cdot\ln 2}} \tag{53}$$
$$= -\frac{a}{\sigma} \cdot e^{-\frac{ln2}{2\cdot\ln 2}} \tag{54}$$
$$= -\frac{a}{\sigma} \cdot e^{-\frac{1}{2}}, \tag{55}$$

and

$$G'_{a,\mu,\sigma}(\mu+\sigma) = -a \cdot \frac{\mu+\sigma-\mu}{\sigma^2} \cdot e^{-\frac{1}{2}\left(\frac{\mu+\sigma-\mu}{\sigma}\right)^2} \tag{56}$$

$$= -\frac{a}{\sigma} \cdot e^{-\frac{1}{2}}. \tag{57}$$

Due to the symmetry of $h_{a,\alpha_0,\beta_0,\mu,\sigma}$ and $G_{a,\mu,\sigma}$ with respect to μ, the equations

$$h_{a,\alpha_0,\beta_0,\mu,\sigma}(\mu-\sigma) = G_{a,\mu,\sigma}(\mu-\sigma), \tag{58}$$

and

$$h'_{a,\alpha_0,\beta_0,\mu,\sigma}(\mu-\sigma) = G'_{a,\mu,\sigma}(\mu-\sigma) \tag{59}$$

hold as well. (Details are left as an exercise.) $\triangle$

Writing the powers in terms of the Euler number e simplifies implementation as computer software in some cases:

Corollary 2 *At $x=\mu$, and at $x=\mu\pm\sigma$, the function*

$$h_{a,\alpha_0,\beta_0,\mu,\sigma}(x) = \begin{cases} a & \text{for } x=0 \\ a \cdot e^{-\frac{1}{2\cdot\ln 2}\cdot\ln\left(e^{4\cdot(\ln 2)\cdot\ln\left(\frac{|x-\mu|}{\sigma}\right)}+1\right)}, & \text{else,} \end{cases} \tag{60}$$

shares function values and first derivatives with

$$G_{a,\mu,\sigma}(x) = a \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}. \tag{61}$$

By applying $A^z = e^{z\ln A}$, write $h_{a,\alpha,\beta,\mu,\sigma}(x)$ as

$$h_{a,\alpha,\beta,\mu,\sigma}(x) = a \cdot \left(\left(\frac{|x-\mu|}{\sigma}\right)^{\alpha}+1\right)^{-\beta} \tag{62}$$

$$= a \cdot e^{-\beta\cdot\ln\left(e^{\alpha\cdot\ln\left(\frac{|x-\mu|}{\sigma}\right)}+1\right)}, \tag{63}$$

and apply it to $\alpha=\alpha_0$, and $\beta=\beta_0$. $\triangle$

Remark 3 *Once a Gauss curve approximating a time series is found, Lemma 1 can be used to find a 0-th approximation within the family of functions of type $h_{a,\alpha,\beta,\mu,\sigma}$.*

Numerical values for α_0 and β_0 are

$$\alpha_0 = 4\cdot\ln 2 = \ln 16 \approx 2.772588722, \tag{64}$$

and

$$\beta_0 = \frac{1}{2\cdot\ln 2} = \frac{1}{\ln 4} \approx 0.7213475205. \tag{65}$$

The function value at $\mu\pm\sigma$ is

$$h_{a,\alpha_0,\beta_0,\mu,\sigma}(\mu\pm\sigma) = G_{a,\mu,\sigma}(\mu\pm\sigma) = a\cdot e^{-\frac{1}{2}} \approx 0.6065306597\ a. \tag{66}$$

The derivatives at $\mu\pm\sigma$ are

$$h'_{a,\alpha_0,\beta_0,\mu,\sigma}(\mu\pm\sigma) = G_{a,\mu,\sigma}(\mu\pm\sigma)' = \mp\frac{a}{\sigma}\cdot e^{-\frac{1}{2}} \approx \mp 0.6065306597\ \frac{a}{\sigma}. \tag{67}$$

3.3 Fourier Series for the Linear Case

3.3.1 Basics

For the intended applications Fourier series of the following type will be an appropriate basis:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cdot \cos(kx) + b_k \cdot \sin(kx)). \tag{68}$$

The parameter a_0 can be calculated by

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \mathrm{d}x. \tag{69}$$

For $k \geq 1$,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) \mathrm{d}x, \tag{70}$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \mathrm{d}x. \tag{71}$$

3.3.2 Auxillary Integrals

Let

$$f(x) = ux + v, \tag{72}$$

with $u, v \in \mathbf{R}$. Let $x_1 < x_2$.

With

$$\begin{aligned} C_0(u, v, x_1, x_2) &:= \int_{x_1}^{x_2} f(x) \mathrm{d}x && (73) \\ &= \int_{x_1}^{x_2} (ux + v) \mathrm{d}x, && (74) \end{aligned}$$

$$\begin{aligned} C_0(u, v, x_1, x_2) &= \left[\frac{ux^2}{2} + vx \right]_{x_1}^{x_2} && (75) \\ &= \frac{ux_2^2}{2} + vx_2 - \frac{ux_1^2}{2} - vx_1. && (76) \end{aligned}$$

Let $1 \leq k \in \mathbf{N}$. Then

$$\begin{aligned} C_k(u, v, x_1, x_2) &:= \int_{x_1}^{x_2} f(x) \cos(kx) \mathrm{d}x && (77) \\ &= \int_{x_1}^{x_2} (ux + v) \cos(kx) \mathrm{d}x. && (78) \end{aligned}$$

By partial integration

$$C_k(u,v,x_1,x_2) = \left[\frac{ux+v}{k}\sin(kx)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{u}{k}\sin(kx)\mathrm{d}x \tag{79}$$
$$= \left[\frac{ux+v}{k}\sin(kx)\right]_{x_1}^{x_2} + \left[\frac{u}{k^2}\cos(kx)\right]_{x_1}^{x_2} \tag{80}$$
$$= \frac{ux_2+v}{k}\sin(kx_2) + \frac{u}{k^2}\cos(kx_2) \tag{81}$$
$$-\frac{ux_1+v}{k}\sin(kx_1) - \frac{u}{k^2}\cos(kx_1). \tag{82}$$

Same with

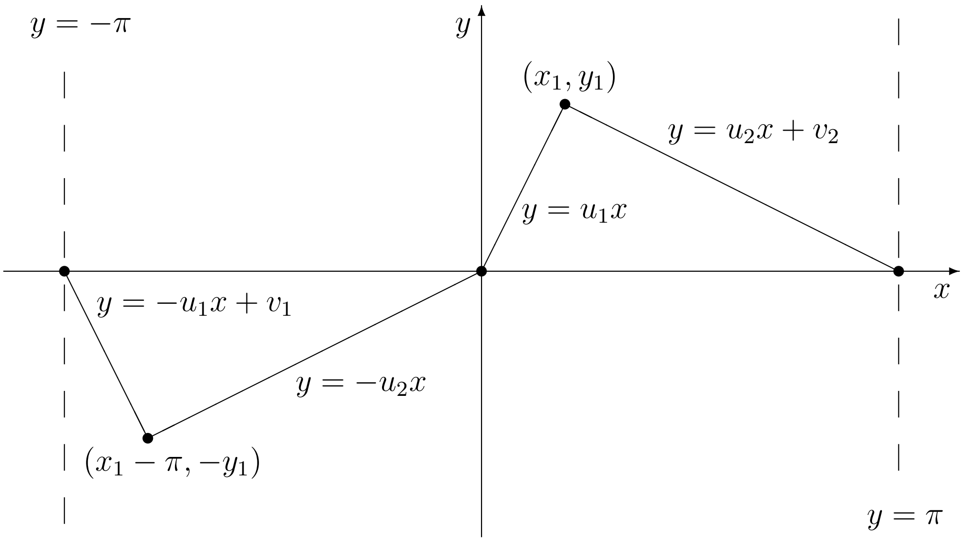
$$S_k(u,v,x_1,x_2) := \int_{x_1}^{x_2} f(x)\sin(kx)\mathrm{d}x \tag{83}$$
$$= \int_{x_1}^{x_2}(ux+v)\sin(kx)\mathrm{d}x. \tag{84}$$

By partial integration

$$S_k(u,v,x_1,x_2) = \left[-\frac{ux+v}{k}\cos(kx)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} -\frac{u}{k}\cos(kx)\mathrm{d}x \tag{85}$$
$$= \left[-\frac{ux+v}{k}\cos(kx)\right]_{x_1}^{x_2} + \left[\frac{u}{k^2}\sin(kx)\right]_{x_1}^{x_2} \tag{86}$$
$$= -\frac{ux_2+v}{k}\cos(kx_2) + \frac{u}{k^2}\sin(kx_2) \tag{87}$$
$$+\frac{ux_1+v}{k}\cos(kx_1) - \frac{u}{k^2}\sin(kx_1). \tag{88}$$

3.3.3 Line Fragments

Think of a distorted sinus function being crudely approximated by a function f composed of four lines per 2π-period:



For $u_1 \neq u_2$, the lines $y = u_1 x$ and $y = u_2 x + v_2$ intersect at (x_1, y_1), with $u_1 x_1 = y_1 = u_2 x_1 + v_2$, hence for

$$v_2 = u_1 x_1 - u_2 x_1 = (u_1 - u_2) x_1, \tag{89}$$

or $x_1 = \frac{v_2}{u_1 - u_2}$, and $y_1 = u_1 x_1 = \frac{u_1 v_2}{u_1 - u_2}$.

With the constraint $u_2 \pi + v_2 = 0$,

$$v_2 = -u_2 \pi. \tag{90}$$

Hence

$$x_1 = \frac{u_2 \pi}{u_2 - u_1}, \tag{91}$$

and

$$y_1 = \frac{u_1 u_2 \pi}{u_2 - u_1}. \tag{92}$$

With the constraint $-u_1 \cdot (-\pi) + v_1 = 0$,

$$v_1 = -u_1 \pi. \tag{93}$$

Note:

$$\forall x \in \mathbf{R} : f(x - \pi) = -f(x). \tag{94}$$

Hence

$$\int_{-\pi}^{0} f(x) \mathrm{d}x = -\int_{0}^{\pi} f(x) \mathrm{d}x, \tag{95}$$

and by the additivity of integrals

$$\int_{-\pi}^{\pi} f(x) \mathrm{d}x = 0. \tag{96}$$

3.3.4 Fourier Coefficients

By equation (96), for f the Fourier coefficient $a_0 = 0$.

For $k \geq 1$,

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) \mathrm{d}x && (97) \\ &= (C_k(-u_1, v_1, -\pi, x_1 - \pi) && (98) \\ &+ C_k(-u_2, 0, x_1 - \pi, 0) && (99) \\ &+ C_k(u_1, 0, 0, x_1) && (100) \\ &+ C_k(u_2, v_2, x_1, \pi))/\pi, && (101) \end{aligned}$$

or, with equations (91), (93), and (90), and considering

$$\frac{u_2 \pi}{u_2 - u_1} - \pi = \frac{u_2 \pi - u_2 \pi + u_1 \pi}{u_2 - u_1} = \frac{u_1 \pi}{u_2 - u_1},$$

$$\begin{aligned}
a_k &= \Big(C_k(-u_1, -u_1\pi, -\pi, \frac{u_1\pi}{u_2-u_1}) && (102)\\
&+ C_k(-u_2, 0, \frac{u_1\pi}{u_2-u_1}, 0) && (103)\\
&+ C_k(u_1, 0, 0, \frac{u_2\pi}{u_2-u_1}) && (104)\\
&+ C_k(u_2, -u_2\pi, \frac{u_2\pi}{u_2-u_1}, \pi)\Big)/\pi. && (105)
\end{aligned}$$

The same way,

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\mathrm{d}x && (106)\\
&= \Big(S_k(-u_1, -u_1\pi, -\pi, \frac{u_1\pi}{u_2-u_1}) && (107)\\
&+ S_k(-u_2, 0, \frac{u_1\pi}{u_2-u_1}, 0) && (108)\\
&+ S_k(u_1, 0, 0, \frac{u_2\pi}{u_2-u_1}) && (109)\\
&+ S_k(u_2, -u_2\pi, \frac{u_2\pi}{u_2-u_1}, \pi)\Big)/\pi. && (110)
\end{aligned}$$

3.4 Smoothing

Modifying equation (68) to

$$\hat{f}(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cdot \cos(kx) + b_k \cdot \sin(kx))\zeta^{-k}, \tag{111}$$

with $\zeta > 1$ returns a smoothed version of the underlying Fourier series, and it is itself a Fourier series. Besides smoothing, this modification makes the result shallower. To account for this effect, and to ensure $f'(0) = 1$, division by the first derivative at $x = 0$ is an approach, if $f'(0)$ is defined. Formally:

$$\hat{f}'(x) = \sum_{k=1}^{\infty}(-a_k \cdot \sin(kx) + b_k \cdot \cos(kx))k\zeta^{-k}. \tag{112}$$

By $\sin(0) = 0$, and $\cos(0) = 1$,

$$\hat{f}'(0) = \sum_{k=1}^{\infty} b_k k\zeta^{-k}, \tag{113}$$

if defined.

A smoothed Fourier series $\tilde{f}$, with $\tilde{f}'(0) = 1$, can hence be derived from the Fourier series (68) either as

$$\begin{aligned}
\tilde{f}(x) &= \frac{\hat{f}(x)}{\sum_{k=1}^{\infty} b_k k\zeta^{-k}} && (114)\\
&= \frac{\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cdot \cos(kx) + b_k \cdot \sin(kx))\zeta^{-k}}{\sum_{k=1}^{\infty} b_k k\zeta^{-k}}, && (115)
\end{aligned}$$

or written formally as Fourier series

$$\tilde{f}(x) = \frac{\tilde{a}_0}{2} + \sum_{k=1}^{\infty}(\tilde{a}_k \cdot \cos(kx) + \tilde{b}_k \cdot \sin(kx)), \tag{116}$$

with

$$\tilde{a}_k = w_k a_k, \tag{117}$$

and

$$\tilde{b}_k = w_k b_k, \tag{118}$$

provided the weights w_k are defined by

$$w_k = \frac{\zeta^{-k}}{\sum_{j=1}^{\infty} b_j j \zeta^{-j}}. \tag{119}$$

3.5 Considering Skewness and Kurtosis

3.5.1 An Easy, But Spurious Approach

Apply equation (9) to equation (111) the following way:

$$\check{h}(x) := h\left(2r\hat{f}\left(\frac{x-\mu}{2r}\right) + \varphi\right), \tag{120}$$

with a scaling factor r, and an offset φ with

$$\varphi := \mu - 2r\hat{f}(0). \tag{121}$$

Although $f(0) = 0$, $\hat{f}(0) \neq 0$, in general. Equation (121) ensures

$$\check{h}(\mu) = h(\mu) \tag{122}$$

by

$$\begin{aligned} \check{h}(\mu) &= h\left(2r\hat{f}\left(\frac{\mu-\mu}{2r}\right) + \varphi\right) && (123)\\ &= h\left(2r\hat{f}\,(0) + \varphi\right) && (124)\\ &= h\left(2r\hat{f}\,(0) + \mu - 2r\hat{f}(0)\right) && (125)\\ &= h\,(\mu)\,. && (126) \end{aligned}$$

Equation (111) can be evaluated at $x = 0$:

$$\hat{f}(0) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cdot \cos(0) + b_k \cdot \sin(0))\zeta^{-k} = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \zeta^{-k}. \tag{127}$$

The scaling factor r is determined by the angluar velocity of the camera, and the frequency the framelet images have been taken. For EFB03, $2\pi r \approx 79$ framelet heights.

This approach undermines the π-periodicity — like in equation (94) — of the absolute amount of the Fourier series. This property, however, is desired.

3.5.2 Saving the π-Quasi-Periodicity

The π-quasi-periodicity of equation (94) is maintained, when the function described by the Fourier series is shifted horizontally (along the x-axis), instead of vertically:

$$\hat{h}(x) := h\left(2r\hat{f}\left(\frac{x-\lambda}{2r}\right)+\mu\right). \tag{128}$$

Determine λ, such that

$$\hat{f}\left(\frac{\mu-\lambda}{2r}\right)=0, \tag{129}$$

or

$$\hat{f}(\xi)=0, \tag{130}$$

with

$$\xi=\frac{\mu-\lambda}{2r}. \tag{131}$$

This way

$$\hat{h}(\mu)=h(\mu) \tag{132}$$

in analogy to equation (122), by

$$\begin{aligned}\hat{h}(\mu) &= h\left(2r\hat{f}\left(\frac{\mu-\lambda}{2r}\right)+\mu\right) && (133)\\ &= h\left(2r\cdot 0+\mu\right) && (134)\\ &= h\left(\mu\right). && (135)\end{aligned}$$

Once ξ is determined, λ is obtained from equation (131) by

$$\lambda=\mu-2r\xi. \tag{136}$$

Equation (128) can hence be written as

$$\hat{h}(x)=h\left(2r\hat{f}\left(\frac{x-\lambda}{2r}\right)+\lambda+2r\xi\right), \tag{137}$$

with equation (9) adjusted to

$$h(x)=a\cdot\left(\left(\frac{\mid x-\lambda-2r\xi\mid}{\varrho}\right)^{\alpha}+1\right)^{-\beta}. \tag{138}$$

In many cases, equation (112) allows approximating a ξ fulfilling equation (130) by applying a 1-dimensional Newton method, starting with $\xi_0=0$, and iterating

$$\xi_{n+1}=\xi_n-\frac{\hat{f}(\xi_n)}{\hat{f}'(\xi_n)}. \tag{139}$$

The existence of a ξ, with $\hat{f}(\xi) = 0$, is necessary; this is not considered as a strong constraint by design of $\hat{f}$, as long as u_1 and $u_2 \neq 0$; in the context of EFB03 investigation, $|u_1|$ and $|u_2| \approx 1$ is to be expected. Applicability depends further on local similarity of $\hat{f}$ to $\iota(x) = x$. In more distorted cases, choosing ξ_0 with

$$|\hat{f}(\xi_0)| < \varepsilon,$$

and $\varepsilon > 0$ sufficiently small, may converge more reliably.

This approach violates $\hat{f}'(\xi) = 1$, in general. But this constraint is less important, and the approach can be adjusted easily by dividing $\hat{f}'$ by $\hat{f}'(\xi)$, if necessary, and if $\hat{f}'(\xi) \neq 0$.

3.5.3 Simplification

Merging equation (137) and (138) to

$$\hat{h}(x) = a \cdot \left(\left(\frac{\mid \left(2r\hat{f}\left(\frac{x-\lambda}{2r}\right) + \lambda + 2r\xi \right) - \lambda - 2r\xi \mid}{\varrho} \right)^{\alpha} + 1 \right)^{-\beta}, \tag{140}$$

and simplification to

$$\hat{h}(x) = a \cdot \left(\left(\frac{\mid 2r\hat{f}\left(\frac{x-\lambda}{2r}\right) \mid}{\varrho} \right)^{\alpha} + 1 \right)^{-\beta} \tag{141}$$

suggests dropping of μ, and using λ as parameter, instead. The somewhat inconvenient determination of ξ is eliminated.

3.6 Determining the Parameters

3.6.1 Basics About Minimizing Square Error Sums

Let $n \in \mathbf{N}$. Let $(c_i)_{i=1}^n$ be a list (time series) of real numbers. Let $(g_i(x_1, \ldots, x_m))_{i=1}^n$, with $m \in \mathbf{N}$, be a family of lists of real numbers, parameterized by the $x_1, \ldots, x_m \in \mathbf{R}$. A best list of real numbers within this family is requested, in the sense of minimizing the square error sum:

$$q(x_1, \ldots, x_m) := \sum_{i=1}^{n} (g_i(x_1, \ldots, x_m) - c_i)^2 \stackrel{!}{=} \min. \tag{142}$$

In many cases the minimum is a local minimum. Such a local minimum can be found in more or less smart ways by descending from some chosen start vector X_0 and looking for another vector X_1, such that $q(X_1) < q(X_0)$. This step is repeated, until no better X_j can be found according to some criteria.

Some of the methods usually applied involve the gradient of q. The gradient is the vector of partial derivatives. Such a partial derivative, say for x_k, $1 \leq k \leq m$, can be calculated the following way:

$$
\begin{aligned}
\frac{\partial q(x_1,\ldots,x_m)}{\partial x_k} &= \frac{\partial(\sum_{i=1}^{n}(g_i(x_1,\ldots,x_m)-c_i)^2)}{\partial x_k} && (143)\\
&= \sum_{i=1}^{n} 2(g_i(x_1,\ldots,x_m)-c_i)\frac{\partial(g_i(x_1,\ldots,x_m)-c_i)}{\partial x_k} && (144)\\
&= 2\sum_{i=1}^{n}(g_i(x_1,\ldots,x_m)-c_i)\frac{\partial g_i(x_1,\ldots,x_m)}{\partial x_k}. && (145)
\end{aligned}
$$

3.6.2 Applicability of Gradient and Quasi-Newton Methods

Estimating the gradient of a function by difference quotients is numerically unstable, since function values almost cancel out for small deltas. Larger deltas can result in high inaccuracies. Therefore working with explicite formulas for the partial derivatives is desirable.

Partial derivatives of $\hat{h}$, defined in equation (141) can be determined explicitely. Relevant are derivatives of $\hat{h}$ by u_1, u_2, ζ, a, b, α, β, ϱ, and λ. The parameter r can probably be determined from telemetry data, or by other visual methods. The parameter u_2 may be defined as $u_2 := -\frac{1}{u_1}$.

Calculating the second derivative (Hessian matrix) of q, needed for the Newton method, is too complex to be considered. But, provided the gradient of q behaves well near the respective equilibrium point, an estimate of the Hessian matrix as a matrix of quotients of differences from gradients at two points may be sufficient for a quasi-Newton method to converge in a linear way.

For better numerical stability, Fourier series, including derivatives, should be summed up in reverse order.

The region of convergence for quasi-Newton methods can be extended by applying the method to $q^l(x_1,\ldots,x_m)$, with $l > 1$, instead of applying it to $q(x_1,\ldots,x_m)$. The needed partial derivatives are calculated by

$$
\frac{\partial q^l(x_1,\ldots,x_m)}{\partial x_k} = l \cdot q^{l-1}(x_1,\ldots,x_m) \cdot \frac{\partial q(x_1,\ldots,x_m)}{\partial x_k}. \qquad (146)
$$

This can slow down the convergence considerably, however. It's therefore more efficient to reduce the exponent l stepwise down to 1, as the method approaches the solution.

3.6.3 Partial Derivatives of $\hat{h}$

After changing the notation of equation (141), to explicitely show the parameters of interest

$$
\hat{h}_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda) = a \cdot \left(\left(\frac{\mid 2r\hat{f}\left(\frac{x-\lambda}{2r}\right)\mid}{\varrho}\right)^{\alpha} + 1\right)^{-\beta}, \qquad (147)
$$

and defining

$$F_x(u_1, u_2, \zeta, \varrho, \lambda) := \frac{\mid 2r\hat{f}\left(\frac{x-\lambda}{2r}\right) \mid}{\varrho}, \tag{148}$$

hence

$$\hat{h}_x(u_1, u_2, \zeta, a, \alpha, \beta, \varrho, \lambda) = a \cdot \left(F_x^\alpha(u_1, u_2, \zeta, \varrho, \lambda) + 1\right)^{-\beta}, \tag{149}$$

here the calculation of the partial derivatives, first with respect to a:

$$\begin{aligned} \frac{\partial \hat{h}_x(u_1, u_2, \zeta, a, \alpha, \beta, \varrho, \lambda)}{\partial a} &= \frac{\partial \left(a \cdot \left(F_x^\alpha(u_1, u_2, \zeta, \varrho, \lambda) + 1\right)^{-\beta}\right)}{\partial a} &(150)\\ &= \left(F_x^\alpha(u_1, u_2, \zeta, \varrho, \lambda) + 1\right)^{-\beta}. &(151)\end{aligned}$$

When deriving with respect to α, consider

$$A^\gamma = e^{\ln A^\gamma} = e^{\gamma \ln A},$$

hence

$$\frac{\partial A^\gamma}{\partial \gamma} = \frac{\partial e^{\gamma \ln A}}{\partial \gamma} = e^{\gamma \ln A} \ln A = A^\gamma \ln A,$$

for $A > 0$:

$$\begin{aligned} &\frac{\partial \hat{h}_x(u_1, u_2, \zeta, a, \alpha, \beta, \varrho, \lambda)}{\partial \alpha} &(152)\\ = &\frac{\partial \left(a \cdot \left(F_x^\alpha(u_1, u_2, \zeta, \varrho, \lambda) + 1\right)^{-\beta}\right)}{\partial \alpha} &(153)\\ = &\frac{-a \cdot F_x^\alpha(u_1, u_2, \zeta, \varrho, \lambda) \ln F_x(u_1, u_2, \zeta, \varrho, \lambda)}{\left(F_x^\alpha(u_1, u_2, \zeta, \varrho, \lambda) + 1\right)^{\beta+1}}. &(154)\end{aligned}$$

Writing $\hat{h}_x(u_1, u_2, \zeta, a, \alpha, \beta, \varrho, \lambda)$ as

$$\hat{h}_x(u_1, u_2, \zeta, a, \alpha, \beta, \varrho, \lambda) = a \cdot \left(\frac{1}{F_x^\alpha(u_1, u_2, \zeta, \varrho, \lambda) + 1}\right)^\beta, \tag{155}$$

simplifies derivation with respect to β:

$$\begin{aligned} &\frac{\partial \hat{h}_x(u_1, u_2, \zeta, a, \alpha, \beta, \varrho, \lambda)}{\partial \alpha} &(156)\\ = &\frac{\partial \left(a \cdot \left(\frac{1}{F_x^\alpha(u_1, u_2, \zeta, \varrho, \lambda)+1}\right)^\beta\right)}{\partial \alpha} &(157)\\ = &\ a \cdot \left(\frac{1}{F_x^\alpha(u_1, u_2, \zeta, \varrho, \lambda) + 1}\right)^\beta \cdot \ln\left(\frac{1}{F_x^\alpha(u_1, u_2, \zeta, \varrho, \lambda) + 1}\right). &(158)\end{aligned}$$

Derivatives of u_1, u_2, ζ, ϱ, and λ share the common factor

$$D_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda) = \frac{-a\cdot\beta\cdot\alpha F_x^{\alpha-1}(u_1,u_2,\zeta,\varrho,\lambda)}{(F_x^{\alpha}(u_1,u_2,\zeta,\varrho,\lambda)+1)^{\beta+1}}. \tag{159}$$

by

$$\begin{aligned}
& D_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda) && (160)\\
:= & \frac{\partial\left(a\cdot\left(\frac{1}{F_x^{\alpha}(u_1,u_2,\zeta,\varrho,\lambda)+1}\right)^{\beta}\right)}{\partial F_x(u_1,u_2,\zeta,\varrho,\lambda)} && (161)\\
= & \frac{a\beta}{(F_x^{\alpha}(u_1,u_2,\zeta,\varrho,\lambda)+1)^{\beta-1}}\cdot\frac{\alpha F_x^{\alpha-1}(u_1,u_2,\zeta,\varrho,\lambda)}{(-F_x^{\alpha}(u_1,u_2,\zeta,\varrho,\lambda)+1)^2} && (162)\\
= & \frac{-a\cdot\beta\cdot\alpha F_x^{\alpha-1}(u_1,u_2,\zeta,\varrho,\lambda)}{(F_x^{\alpha}(u_1,u_2,\zeta,\varrho,\lambda)+1)^{\beta+1}}. && (163)
\end{aligned}$$

due to the chain rule.

Deriving with respect to ϱ: First consider

$$\begin{aligned}
\frac{\partial F_x(u_1,u_2,\zeta,\varrho,\lambda)}{\partial\varrho} &= \frac{\partial\frac{|2r\hat{f}\left(\frac{x-\lambda}{2r}\right)|}{\varrho}}{\partial\varrho} && (164)\\
&= -\frac{|\;2r\hat{f}\left(\frac{x-\lambda}{2r}\right)\;|}{\varrho^2} && (165)\\
&= -\frac{F_x(u_1,u_2,\zeta,\varrho,\lambda)}{\varrho}. && (166)
\end{aligned}$$

Then

$$\begin{aligned}
& \frac{\partial\hat{h}_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)}{\partial\varrho} && (167)\\
= & \frac{\partial\left(a\cdot\left(\frac{1}{F_x^{\alpha}(u_1,u_2,\zeta,\varrho,\lambda)+1}\right)^{\beta}\right)}{\partial\varrho} && (168)\\
= & D_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)\cdot\frac{\partial F_x(u_1,u_2,\zeta,\varrho,\lambda)}{\partial\varrho} && (169)\\
= & -D_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)\cdot\frac{F_x(u_1,u_2,\zeta,\varrho,\lambda)}{\varrho}. && (170)
\end{aligned}$$

Deriving with respect to λ, for $x\neq\lambda$:

$$\begin{aligned}
& \frac{\partial\hat{h}_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)}{\partial\lambda} && (171)\\
= & \frac{\partial\left(a\cdot\left(\frac{1}{F_x^{\alpha}(u_1,u_2,\zeta,\varrho,\lambda)+1}\right)^{\beta}\right)}{\partial\lambda} && (172)
\end{aligned}$$

$$
\begin{aligned}
&= D_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)\cdot\frac{\partial F_x(u_1,u_2,\zeta,\varrho,\lambda)}{\partial\lambda} && (173)\\
&= D_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)\cdot\frac{\partial\frac{|2r\hat{f}\left(\frac{x-\lambda}{2r}\right)|}{\varrho}}{\partial\lambda} && (174)\\
&= D_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)\cdot\frac{2r}{\varrho}\cdot\frac{\partial\mid\hat{f}\left(\frac{x-\lambda}{2r}\right)\mid}{\partial\lambda} && (175)\\
&= D_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)\cdot\frac{2r}{\varrho}\cdot\mathrm{sgn}\hat{f}\left(\frac{x-\lambda}{2r}\right)\cdot\frac{\partial\hat{f}\left(\frac{x-\lambda}{2r}\right)}{\partial\lambda} && (176)\\
&= D_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)\cdot\frac{2r}{\varrho}\cdot\mathrm{sgn}\hat{f}\left(\frac{x-\lambda}{2r}\right)\cdot\hat{f}'\left(\frac{x-\lambda}{2r}\right)\cdot\frac{-1}{2r} && (177)\\
&= -\frac{D_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)}{\varrho}\cdot\mathrm{sgn}\hat{f}\left(\frac{x-\lambda}{2r}\right)\cdot\hat{f}'\left(\frac{x-\lambda}{2r}\right). && (178)
\end{aligned}
$$

For $\hat{f}'$, see equation (112).

Continue with the chain rule to get access to the argument of $\hat{f}$, for $\varrho, r > 0$

$$
\frac{\partial\frac{|2r\hat{f}\left(\frac{x-\lambda}{2r}\right)|}{\varrho}}{\partial\frac{x-\lambda}{2r}} = \frac{2r}{\varrho}\cdot\mathrm{sgn}\hat{f}\left(\frac{x-\lambda}{2r}\right), \tag{179}
$$

and define

$$
E_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda) := D_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)\cdot\frac{2r}{\varrho}\cdot\mathrm{sgn}\hat{f}\left(\frac{x-\lambda}{2r}\right). \tag{180}
$$

Then

$$
\begin{aligned}
\frac{\partial\hat{h}_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)}{\partial u_1} &= E_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)\cdot\frac{\partial\hat{f}\left(\frac{x-\lambda}{2r}\right)}{\partial u_1}, && (181)\\
\frac{\partial\hat{h}_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)}{\partial u_2} &= E_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)\cdot\frac{\partial\hat{f}\left(\frac{x-\lambda}{2r}\right)}{\partial u_2}, && (182)\\
\frac{\partial\hat{h}_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)}{\partial\zeta} &= E_x(u_1,u_2,\zeta,a,\alpha,\beta,\varrho,\lambda)\cdot\frac{\partial\hat{f}\left(\frac{x-\lambda}{2r}\right)}{\partial\zeta}. && (183)
\end{aligned}
$$

3.6.4 Partial Derivatives of $\hat{f}$ for $\zeta = 1$

Fourier series don't converge well near discontinuities. The better (and faster) way to determine partial derivatives of $\hat{f}$ for $\zeta = 1$ is hence a treatment in terms of linear functions. Assume $u_2 = u_2(u_1) = -\frac{1}{u_1}$.

For $\pi \le x \le \frac{u_2\pi}{u_2-u1}$,

$$
\hat{f}(x,u_1) = u_1 x, \tag{184}
$$

$$\frac{\partial \hat{f}(x,u_1)}{x} = u_1, \tag{185}$$

$$\frac{\partial \hat{f}(x,u_1)}{u_1} = x. \tag{186}$$

For $\frac{u_2\pi}{u_2-u1} \leq x \leq \pi$,

$$\hat{f}(x,u_1) = u_2 \cdot (x-\pi), \tag{187}$$

$$\frac{\partial \hat{f}(x,u_1)}{x} = u_2, \tag{188}$$

$$\frac{\partial \hat{f}(x,u_1)}{u_1} = \frac{x-\pi}{u_1^2}. \tag{189}$$

For $-\pi \leq x \leq \frac{u_2\pi}{u_2-u1} - \pi$,

$$\hat{f}(x,u_1) = -u_1 \cdot (x+\pi), \tag{190}$$

$$\frac{\partial \hat{f}(x,u_1)}{x} = -u_1, \tag{191}$$

$$\frac{\partial \hat{f}(x,u_1)}{u_1} = -(x+\pi). \tag{192}$$

For $\frac{u_2\pi}{u_2-u1} - \pi \leq x \leq 0$,

$$\hat{f}(x,u_1) = -u_2 x, \tag{193}$$

$$\frac{\partial \hat{f}(x,u_1)}{x} = -u_2, \tag{194}$$

$$\frac{\partial \hat{f}(x,u_1)}{u_1} = -\frac{x}{u_1^2}. \tag{195}$$

For $x \notin [-\pi;\pi]$, use $\hat{f}(x+2z\pi, u_1) = \hat{f}(x,u_1)$, for integers $z \in \mathbf{Z}$.

3.7 For Completeness, the Remaining Partial Derivatives

Partial derivatives of this subsection behave instable in many cases. This applies especially to difference quotients, when used as an approximation for Hessian matrices.

3.7.1 Partial Derivatives of $\hat{f}$ for $\zeta \neq 1$

Write equation (111) as

$$\hat{f}_x(u_1,u_2,\zeta) = \frac{a_0(u_1,u_2)}{2} + \sum_{k=1}^{\infty} (a_k(u_1,u_2)\cdot\cos(kx) + b_k(u_1,u_2)\cdot\sin(kx))\zeta^{-k}, \tag{196}$$

to emphasize the dependence of $\hat{f}$ on u_1, u_2, and ζ.

Derive with respect to ζ:

$$\frac{\partial \hat{f}_x(u_1, u_2, \zeta)}{\partial \zeta} \tag{197}$$

$$= \frac{\partial \left[\frac{a_0(u_1,u_2)}{2} + \sum_{k=1}^{\infty}(a_k(u_1,u_2)\cdot\cos(kx) + b_k(u_1,u_2)\cdot\sin(kx))\zeta^{-k}\right]}{\partial \zeta} \tag{198}$$

$$= -\sum_{k=1}^{\infty}(a_k(u_1,u_2)\cdot\cos(kx) + b_k(u_1,u_2)\cdot\sin(kx))k\zeta^{-k-1}. \tag{199}$$

Deriving with respect to u_1

$$\frac{\partial \hat{f}_x(u_1, u_2, \zeta)}{\partial u_1} \tag{200}$$

$$= \frac{\partial \left[\frac{a_0(u_1,u_2)}{2} + \sum_{k=1}^{\infty}(a_k(u_1,u_2)\cdot\cos(kx) + b_k(u_1,u_2)\cdot\sin(kx))\zeta^{-k}\right]}{\partial u_1} \tag{201}$$

$$= \frac{1}{2}\cdot\frac{\partial a_0(u_1,u_2)}{\partial u_1} + \sum_{k=1}^{\infty}\left(\frac{\partial a_k(u_1,u_2)}{\partial u_1}\cdot\cos(kx) + \frac{\partial b_k(u_1,u_2)}{\partial u_1}\cdot\sin(kx)\right)\zeta^{-k}, \tag{202}$$

and the same way with respect to u_2

$$\frac{\partial \hat{f}_x(u_1, u_2, \zeta)}{\partial u_2} \tag{203}$$

$$= \frac{1}{2}\cdot\frac{\partial a_0(u_1,u_2)}{\partial u_2} + \sum_{k=1}^{\infty}\left(\frac{\partial a_k(u_1,u_2)}{\partial u_2}\cdot\cos(kx) + \frac{\partial b_k(u_1,u_2)}{\partial u_2}\cdot\sin(kx)\right)\zeta^{-k}, \tag{204}$$

requires calculation of $\frac{\partial a_k(u_1,u_2)}{\partial u_1}$, $\frac{\partial b_k(u_1,u_2)}{\partial u_1}$, $\frac{\partial a_k(u_1,u_2)}{\partial u_2}$, and $\frac{\partial b_k(u_1,u_2)}{\partial u_2}$.

3.7.2 Partial Derivatives of C_k and S_k

The partial derivatives of C_k and S_k will be needed to calculate the partial derivatives of a_k and b_k. First the four partial derivatives of C_k. Case $k = 0$ needs to be treated separately:

$$\frac{\partial C_0(u,v,x_1,x_2)}{\partial u} = \frac{\partial \frac{ux_2^2}{2} + vx_2 - \frac{ux_1^2}{2} - vx_1}{\partial u} \tag{205}$$

$$= \frac{x_2^2}{2} - \frac{x_1^2}{2} \tag{206}$$

$$= \frac{x_2^2 - x_1^2}{2}. \tag{207}$$

$$\frac{\partial C_0(u,v,x_1,x_2)}{\partial v} = \frac{\partial \frac{ux_2^2}{2} + vx_2 - \frac{ux_1^2}{2} - vx_1}{\partial v} \tag{208}$$

$$= x_2 - x_1. \tag{209}$$

$$\begin{aligned}\frac{\partial C_0(u,v,x_1,x_2)}{\partial x_1} &= \frac{\partial \frac{ux_2^2}{2} + vx_2 - \frac{ux_1^2}{2} - vx_1}{\partial x_1} && (210)\\ &= -ux_1 - v. && (211)\end{aligned}$$

$$\begin{aligned}\frac{\partial C_0(u,v,x_1,x_2)}{\partial x_2} &= \frac{\partial \frac{ux_2^2}{2} + vx_2 - \frac{ux_1^2}{2} - vx_1}{\partial x_2} && (212)\\ &= ux_2 + v. && (213)\end{aligned}$$

Now let $k \geq 1$.

$$\begin{aligned}\frac{\partial C_k(u,v,x_1,x_2)}{\partial u} &= \frac{\partial\left[\frac{ux_2+v}{k}\sin(kx_2)\right]}{\partial u} + \frac{\partial\left[\frac{u}{k^2}\cos(kx_2)\right]}{\partial u} && (214)\\ &\quad -\frac{\partial\left[\frac{ux_1+v}{k}\sin(kx_1)\right]}{\partial u} - \frac{\partial\left[\frac{u}{k^2}\cos(kx_1)\right]}{\partial u} && (215)\\ &= \frac{x_2}{k}\sin(kx_2) + \frac{1}{k^2}\cos(kx_2) && (216)\\ &\quad -\frac{x_1}{k}\sin(kx_1) - \frac{1}{k^2}\cos(kx_1). && (217)\end{aligned}$$

$$\begin{aligned}\frac{\partial C_k(u,v,x_1,x_2)}{\partial v} &= \frac{\partial\left[\frac{ux_2+v}{k}\sin(kx_2)\right]}{\partial v} + \frac{\partial\left[\frac{u}{k^2}\cos(kx_2)\right]}{\partial v} && (218)\\ &\quad -\frac{\partial\left[\frac{ux_1+v}{k}\sin(kx_1)\right]}{\partial v} - \frac{\partial\left[\frac{u}{k^2}\cos(kx_1)\right]}{\partial v} && (219)\\ &= \frac{1}{k}\sin(kx_2) - \frac{1}{k}\sin(kx_1). && (220)\end{aligned}$$

$$\begin{aligned}\frac{\partial C_k(u,v,x_1,x_2)}{\partial x_1} &= \frac{\partial\left[\frac{ux_2+v}{k}\sin(kx_2)\right]}{\partial x_1} + \frac{\partial\left[\frac{u}{k^2}\cos(kx_2)\right]}{\partial x_1} && (221)\\ &\quad -\frac{\partial\left[\frac{ux_1+v}{k}\sin(kx_1)\right]}{\partial x_1} - \frac{\partial\left[\frac{u}{k^2}\cos(kx_1)\right]}{\partial x_1} && (222)\\ &= -\frac{u}{k}\sin(kx_1) - (ux_1+v)\cos(kx_1) + \frac{u}{k}\sin(kx_1) && (223)\\ &= -(ux_1+v)\cos(kx_1). && (224)\end{aligned}$$

The same way

$$\frac{\partial C_k(u,v,x_1,x_2)}{\partial x_2} = (ux_2+v)\cos(kx_2). \quad (225)$$

The four partial derivatives of S_k, $k \geq 1$:

$$\frac{\partial S_k(u,v,x_1,x_2)}{\partial u} = -\frac{\partial\left[\frac{ux_2+v}{k}\cos(kx_2)\right]}{\partial u} + \frac{\partial\left[\frac{u}{k^2}\sin(kx_2)\right]}{\partial u} \quad (226)$$

$$
\begin{aligned}
& +\frac{\partial\left[\frac{ux_1+v}{k}\cos(kx_1)\right]}{\partial u}-\frac{\partial\left[\frac{u}{k^2}\sin(kx_1)\right]}{\partial u} && (227)\\
&= -\frac{x_2}{k}\cos(kx_2)+\frac{1}{k^2}\sin(kx_2) && (228)\\
& +\frac{x_1}{k}\cos(kx_1)-\frac{1}{k^2}\sin(kx_1). && (229)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial S_k(u,v,x_1,x_2)}{\partial v} &= -\frac{\partial\left[\frac{ux_2+v}{k}\cos(kx_2)\right]}{\partial v}+\frac{\partial\left[\frac{u}{k^2}\sin(kx_2)\right]}{\partial v} && (230)\\
& +\frac{\partial\left[\frac{ux_1+v}{k}\cos(kx_1)\right]}{\partial v}-\frac{\partial\left[\frac{u}{k^2}\sin(kx_1)\right]}{\partial v} && (231)\\
&= -\frac{1}{k}\cos(kx_2)+\frac{1}{k}\cos(kx_1). && (232)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial S_k(u,v,x_1,x_2)}{\partial x_1} &= -\frac{\partial\left[\frac{ux_2+v}{k}\cos(kx_2)\right]}{\partial x_1}+\frac{\partial\left[\frac{u}{k^2}\sin(kx_2)\right]}{\partial x_1} && (233)\\
& +\frac{\partial\left[\frac{ux_1+v}{k}\cos(kx_1)\right]}{\partial x_1}-\frac{\partial\left[\frac{u}{k^2}\sin(kx_1)\right]}{\partial x_1} && (234)\\
&= \frac{u}{k}\cos(kx_1)-(ux_1+v)\sin(kx_1)-\frac{u}{k}\cos(kx_1) && (235)\\
&= -(ux_1+v)\sin(kx_1). && (236)
\end{aligned}
$$

Again the same way

$$\frac{\partial S_k(u,v,x_1,x_2)}{\partial x_2}=(ux_2+v)\sin(kx_2). \qquad (237)$$

3.7.3 Partial Derivatives of a_k and b_k

With the formulas for a_k and b_k of 3.3.4, the partial derivatives of a_k and b_k can be calculated:

$$
\begin{aligned}
\frac{\partial a_k(u_1,u_2)}{\partial u_1} &= \Bigg(\frac{\partial C_k(-u_1,-u_1\pi,-\pi,\frac{u_1\pi}{u_2-u_1})}{\partial u_1} && (238)\\
&+ \frac{\partial C_k(-u_2,0,\frac{u_1\pi}{u_2-u_1},0)}{\partial u_1} && (239)\\
&+ \frac{\partial C_k(u_1,0,0,\frac{u_2\pi}{u_2-u_1})}{\partial u_1} && (240)\\
&+ \frac{\partial C_k(u_2,-u_2\pi,\frac{u_2\pi}{u_2-u_1},\pi)}{\partial u_1}\Bigg)\cdot\frac{1}{\pi}, && (241)
\end{aligned}
$$

$$
\frac{\partial b_k(u_1,u_2)}{\partial u_1} = \Bigg(\frac{\partial S_k(-u_1,-u_1\pi,-\pi,\frac{u_1\pi}{u_2-u_1})}{\partial u_1} \qquad (242)
$$

$$+ \quad \frac{\partial S_k(-u_2, 0, \frac{u_1\pi}{u_2-u_1}, 0)}{\partial u_1} \tag{243}$$

$$+ \quad \frac{\partial S_k(u_1, 0, 0, \frac{u_2\pi}{u_2-u_1})}{\partial u_1} \tag{244}$$

$$+ \quad \left. \frac{\partial S_k(u_2, -u_2\pi, \frac{u_2\pi}{u_2-u_1}, \pi)}{\partial u_1} \right) \cdot \frac{1}{\pi}, \tag{245}$$

$$\frac{\partial a_k(u_1,u_2)}{\partial u_2} = \left(\frac{\partial C_k(-u_1, -u_1\pi, -\pi, \frac{u_1\pi}{u_2-u_1})}{\partial u_2} \right. \tag{246}$$

$$+ \quad \frac{\partial C_k(-u_2, 0, \frac{u_1\pi}{u_2-u_1}, 0)}{\partial u_2} \tag{247}$$

$$+ \quad \frac{\partial C_k(u_1, 0, 0, \frac{u_2\pi}{u_2-u_1})}{\partial u_2} \tag{248}$$

$$+ \quad \left. \frac{\partial C_k(u_2, -u_2\pi, \frac{u_2\pi}{u_2-u_1}, \pi)}{\partial u_2} \right) \cdot \frac{1}{\pi}, \tag{249}$$

and

$$\frac{\partial b_k(u_1,u_2)}{\partial u_2} = \left(\frac{\partial S_k(-u_1, -u_1\pi, -\pi, \frac{u_1\pi}{u_2-u_1})}{\partial u_2} \right. \tag{250}$$

$$+ \quad \frac{\partial S_k(-u_2, 0, \frac{u_1\pi}{u_2-u_1}, 0)}{\partial u_2} \tag{251}$$

$$+ \quad \frac{\partial S_k(u_1, 0, 0, \frac{u_2\pi}{u_2-u_1})}{\partial u_2} \tag{252}$$

$$+ \quad \left. \frac{\partial S_k(u_2, -u_2\pi, \frac{u_2\pi}{u_2-u_1}, \pi)}{\partial u_2} \right) \cdot \frac{1}{\pi}. \tag{253}$$

With chain rules of the structure

$$\frac{\partial C_k(u(u_1,u_2), v(u_1,u_2), x_1(u_1,u_2), x_2(u_1,u_2)}{\partial u_i} \tag{254}$$

$$= \quad \frac{\partial C_k(u(u_1,u_2), v(u_1,u_2), x_1(u_1,u_2), x_2(u_1,u_2))}{\partial u(u_1,u_2)} \cdot \frac{\partial u(u_1,u_2)}{\partial u_i} \tag{255}$$

$$+ \quad \frac{\partial C_k(u(u_1,u_2), v(u_1,u_2), x_1(u_1,u_2), x_2(u_1,u_2))}{\partial v(u_1,u_2)} \cdot \frac{\partial v(u_1,u_2)}{\partial u_i} \tag{256}$$

$$+ \quad \frac{\partial C_k(u(u_1,u_2), v(u_1,u_2), x_1(u_1,u_2), x_2(u_1,u_2))}{\partial x_1(u_1,u_2)} \cdot \frac{\partial x_1(u_1,u_2)}{\partial u_i} \tag{257}$$

$$+ \quad \frac{\partial C_k(u(u_1,u_2), v(u_1,u_2), x_1(u_1,u_2), x_2(u_1,u_2))}{\partial x_2(u_1,u_2)} \cdot \frac{\partial x_2(u_1,u_2)}{\partial u_i}, \tag{258}$$

$i = 1, 2$, same for S_k, the calculation of a_k and b_k is reduced to the respective calculations of $\frac{\partial u(u_1,u_2)}{\partial u_i}$, $\frac{\partial v(u_1,u_2)}{\partial u_i}$, $\frac{\partial x_1(u_1,u_2)}{\partial u_i}$, and $\frac{\partial x_2(u_1,u_2)}{\partial u_i}$.

The following versions of potentially non-constant $u(u_1, u_2)$, $v(u_1, u_2)$, $x_1(u_1, u_2)$, and $x_2(u_1, u_2)$ occur: u_1, $-u_1$, $-u_1\pi$, u_2, $-u_2$, $-u_2\pi$, $\frac{u_1\pi}{u_2-u_1}$, and $\frac{u_2\pi}{u_2-u_1}$.

Non-zero derivatives:

$$\frac{\partial u_i}{\partial u_i} = 1,$$

$$\frac{\partial(-u_i)}{\partial u_i} = -1,$$

$$\frac{\partial(-u_i\pi)}{\partial u_i} = -\pi,$$

for $i = 1, 2$.

$$\frac{\partial \frac{u_1\pi}{u_2-u_1}}{\partial u_1} = \frac{\pi}{u_2 - u_1} + \frac{u_1\pi}{(u_2 - u_1)^2} = \frac{u_2\pi}{(u_2 - u_1)^2},$$

$$\frac{\partial \frac{u_2\pi}{u_2-u_1}}{\partial u_1} = \frac{u_2\pi}{(u_2 - u_1)^2},$$

$$\frac{\partial \frac{u_1\pi}{u_2-u_1}}{\partial u_2} = -\frac{u_1\pi}{(u_2 - u_1)^2},$$

and

$$\frac{\partial \frac{u_2\pi}{u_2-u_1}}{\partial u_2} = \frac{\pi}{u_2 - u_1} - \frac{u_2\pi}{(u_2 - u_1)^2} = -\frac{u_1\pi}{(u_2 - u_1)^2}.$$

When defining

$$u_2 := -\frac{1}{u_1},$$

the following non-zero cases need to be considered: u_1, $-u_1$, $-u_1\pi$, $-\frac{1}{u_1}$, $\frac{1}{u_1}$, $\frac{\pi}{u_1}$,

$$\frac{u_1\pi}{-\frac{1}{u_1} - u_1} = \frac{-u_1^2\pi}{1 + u_1^2},$$

and

$$\frac{-\frac{1}{u_1}\pi}{-\frac{1}{u_1} - u_1} = \frac{\pi}{1 + u_1^2}.$$

The non-zero derivatives, besides those already calculated:

$$\frac{\partial \left[-\frac{1}{u_1}\right]}{\partial u_1} = \frac{1}{u_1^2}$$

$$\frac{\partial \frac{1}{u_1}}{\partial u_1} = -\frac{1}{u_1^2}$$

$$\frac{\partial \left[\frac{\pi}{u_1}\right]}{\partial u_1} = -\frac{\pi}{u_1^2}$$

$$\frac{\partial \frac{-u_1^2\pi}{1+u_1^2}}{\partial u_1} = \frac{-2u_1\pi\cdot(1+u_1^2)+u_1^2\pi\cdot(2u_1)}{(1+u_1^2)^2} = \frac{-2u_1\pi-2u_1^3\pi+2u_1^3\pi}{(1+u_1^2)^2} = -\frac{2u_1\pi}{(1+u_1^2)^2},$$

and

$$\frac{\partial \frac{\pi}{1+u_1^2}}{\partial u_1} = -\frac{2u_1\pi}{(1+u_1^2)^2}.$$

More explicitely, for the version $u_2 = u_2(u_1) := -\frac{1}{u_1}$, applying the chain rule to

$$\frac{\partial a_k(u_1,u_2(u_1))}{\partial u_1} = \left[\frac{\partial C_k\left(-u_1,-u_1\pi,-\pi,\frac{u_1\pi}{u_2(u_1)-u_1}\right)}{\partial u_1}\right. \tag{259}$$

$$+ \frac{\partial C_k\left(-u_2(u_1),0,\frac{u_1\pi}{u_2(u_1)-u_1},0\right)}{\partial u_1} \tag{260}$$

$$+ \frac{\partial C_k\left(u_1,0,0,\frac{u_2(u_1)\pi}{u_2(u_1)-u_1}\right)}{\partial u_1} \tag{261}$$

$$+ \left.\frac{\partial C_k\left(u_2(u_1),-u_2(u_1)\pi,\frac{u_2(u_1)\pi}{u_2(u_1)-u_1},\pi\right)}{\partial u_1}\right]\cdot\frac{1}{\pi} \tag{262}$$

$$\tag{263}$$

results in

$$\frac{\partial a_k(u_1,u_2(u_1))}{\partial u_1} = \frac{1}{\pi}\cdot\left[-\frac{\partial C_k\left(-u_1,-u_1\pi,-\pi,\frac{u_1\pi}{u_2(u_1)-u_1}\right)}{\partial\left(-u_1\right)}\right. \tag{264}$$

$$- \frac{\partial C_k\left(-u_1,-u_1\pi,-\pi,\frac{u_1\pi}{u_2(u_1)-u_1}\right)}{\partial\left(-u_1\pi\right)}\cdot\pi \tag{265}$$

$$- \frac{\partial C_k\left(-u_1,-u_1\pi,-\pi,\frac{u_1\pi}{u_2(u_1)-u_1}\right)}{\partial\frac{u_1\pi}{u_2(u_1)-u_1}}\cdot\frac{2\pi u_1}{(1+u_1^2)^2} \tag{266}$$

$$- \frac{\partial C_k\left(-u_2(u_1),0,\frac{u_1\pi}{u_2(u_1)-u_1},0\right)}{\partial\left(-u_2(u_1)\right)}\cdot\frac{1}{u_1^2} \tag{267}$$

$$- \frac{\partial C_k\left(-u_2(u_1),0,\frac{u_1\pi}{u_2(u_1)-u_1},0\right)}{\partial\frac{u_1\pi}{u_2(u_1)-u_1}}\cdot\frac{2\pi u_1}{(1+u_1^2)^2} \tag{268}$$

$$+ \frac{\partial C_k\left(u_1,0,0,\frac{u_2(u_1)\pi}{u_2(u_1)-u_1}\right)}{\partial u_1} \tag{269}$$

$$- \frac{\partial C_k\left(u_1,0,0,\frac{u_2(u_1)\pi}{u_2(u_1)-u_1}\right)}{\partial\frac{u_2(u_1)\pi}{u_2(u_1)-u_1}}\cdot\frac{2\pi u_1}{(1+u_1^2)^2} \tag{270}$$

$$+ \frac{\partial C_k\left(u_2(u_1),-u_2(u_1)\pi,\frac{u_2(u_1)\pi}{u_2(u_1)-u_1},\pi\right)}{\partial u_2(u_1)}\cdot\frac{1}{u_1^2} \tag{271}$$

$$- \quad \frac{\partial C_k\left(u_2(u_1), -u_2(u_1)\pi, \frac{u_2(u_1)\pi}{u_2(u_1)-u_1}, \pi\right)}{\partial\left(-u_2(u_1)\pi\right)} \cdot \frac{\pi}{u_1^2} \tag{272}$$

$$- \quad \frac{\partial C_k\left(u_2(u_1), -u_2(u_1)\pi, \frac{u_2(u_1)\pi}{u_2(u_1)-u_1}, \pi\right)}{\partial \frac{u_2(u_1)\pi}{u_2(u_1)-u_1}} \cdot \frac{2\pi u_1}{(1+u_1^2)^2}\Bigg]. \tag{273}$$

Replace C_k by S_k to obtain the formula for $\frac{\partial b_k(u_1)}{\partial u_1} := \frac{\partial b_k(u_1, u_2(u_1))}{\partial u_1}$.

3.8 Application to EFB03

First tests approximating EFB03 data by functions of the $\hat{h}$ family, with $\zeta := 1$, and $u_1 := 1$, showed a significant improvement relative to approximations by Gauss functions.

Additional optimizing of u_1 resulted in further improvement. This is quantified in the tables of appendix B. While u_1 is close to 1.0 on the left of the image, it indicates increasing skewness towards the right side of EFB03 with values for u_1 between about 1.2 and about 1.45. The root mean square error for the approximated time series is below 2% relative to the peak height a for each tested vertical EFB03 stripe of width 100 pixels. (The time series has been a result of linear regression of the horizontal substripe grey values of EFB03, as described in *"Some Empirical Approximations of the Stray Light in Junocam Image EFB03, Part I"*).

The parameters α and β indicating kurtosis, show a considerable variability, as well. This variability doesn't appear to be random. The parameter β decreases overall from left to right. The parameter α appears to increase from left to right, less distinctly, however.

With u_1 diverging from 1.0 to the right side of EFB03, the parameter ζ is expected to get an increasing meaning. It needs a careful treatment due to its computationally challenging definition by a Fourier series.

4 A Family of Generalized Gauss Functions

4.1 First Generalization Step

4.1.1 The Function

The formula

$$G_{a,\mu,\sigma}(x) = a \cdot e^{-\frac{1}{2}\cdot\left(\frac{x-\mu}{\sigma}\right)^2},$$

defining the Gauss functions, can be re-written as

$$\eta_{a,\mu,\sigma,\alpha,\beta}(x) = a \cdot e^{\beta\cdot\left|\frac{x-\mu}{\sigma}\right|^\alpha}, \tag{274}$$

with $\alpha = 2$ and $\beta = -\frac{1}{2}$.

When investigating the other parameters of η, the notation

$$\eta_x(a,\mu,\sigma,\alpha,\beta) = a \cdot e^{\beta\cdot\left|\frac{x-\mu}{\sigma}\right|^\alpha} \tag{275}$$

may be more suggestive.

4.1.2 Partial Derivatives

For the partial derivatives with respect to x, μ, and σ, it's convenient to define a simplified version of η by

$$\vartheta_{a,\alpha,\beta}(x) := a \cdot e^{\beta\cdot|x|^\alpha}. \tag{276}$$

Applying the chain rule to equation (276) returns

$$\vartheta'_{a,\alpha,\beta}(x) = \vartheta_{a,\alpha,\beta}(x) \cdot \alpha\beta \cdot |x|^{\alpha-1} \cdot \operatorname{sgn}(x). \tag{277}$$

Write η in terms of ϑ as

$$\eta_{a,\mu,\sigma,\alpha,\beta}(x) = \eta_x(a,\mu,\sigma,\alpha,\beta) = \vartheta_{a,\alpha,\beta}\left(\frac{x-\mu}{\sigma}\right). \tag{278}$$

Applying the chain rule to equation (278) then returns

$$\eta'_{a,\mu,\sigma,\alpha,\beta}(x) = \vartheta'_{a,\alpha,\beta}\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}, \tag{279}$$

and applying the chain rule to equation (275) returns

$$\frac{\partial\eta_x(a,\mu,\sigma,\alpha,\beta)}{\partial a} = e^{\beta\cdot\left|\frac{x-\mu}{\sigma}\right|^\alpha}, \tag{280}$$

$$\frac{\partial\eta_x(a,\mu,\sigma,\alpha,\beta)}{\partial \mu} = \vartheta'_{a,\alpha,\beta}\left(\frac{x-\mu}{\sigma}\right) \cdot \left(-\frac{1}{\sigma}\right), \tag{281}$$

$$\frac{\partial\eta_x(a,\mu,\sigma,\alpha,\beta)}{\partial \sigma} = \vartheta'_{a,\alpha,\beta}\left(\frac{x-\mu}{\sigma}\right) \cdot \left(-\frac{x-\mu}{\sigma^2}\right), \tag{282}$$

$$\frac{\partial\eta_x(a,\mu,\sigma,\alpha,\beta)}{\partial \alpha} = \eta_x(a,\mu,\sigma,\alpha,\beta) \cdot \beta \cdot \left|\frac{x-\mu}{\sigma}\right|^\alpha \cdot \ln\alpha, \tag{283}$$

and

$$\frac{\partial\eta_x(a,\mu,\sigma,\alpha,\beta)}{\partial \beta} = \eta_x(a,\mu,\sigma,\alpha,\beta) \cdot \left|\frac{x-\mu}{\sigma}\right|^\alpha. \tag{284}$$

4.1.3 Dependence of parameters

Since

$$\begin{aligned} \beta \cdot \left|\frac{x-\mu}{\sigma}\right|^\alpha &= \frac{\beta}{|\sigma|^\alpha} \cdot |x-\mu|^\alpha && (285) \\ &= \beta_1 \cdot |x-\mu|^\alpha , && (286) \end{aligned}$$

with

$$\beta_1 := \frac{\beta}{|\sigma|^\alpha}, \tag{287}$$

the parameters σ and β depend on each other. Hence one of these two parameters can be chosen as a constant, when fitting the parameters of η to a given set of data.

5 Future Research

First attempts to describe the horizontal variability of Gauss functions approximating the EFB03 stray light didn't reveal an obvious canonical approach. This attempt is worth to be repeated with the $\hat{h}$ family of functions. The first goal is to find one or more 2-dimensional functions approximating the EFB03 stray light, with as few parameters as reasonably possible. The next goal is a stepwise extension of this 2d-function to a function with predictive power for any stray light detected by Junocam. Those functions might be related to convolutions.

Part I of this small series of articles intended to include a two-dimensional approximation of EFB03 within part II. But due to the complexity of the topic, it appears appropriate to treat it in a separate article.

Other obvious questions are about the role of the parameter ζ for improving the approximations, and verification (or falsification) of the conjecture, that the $\hat{h}$ family of functions is better-suited to describe EFB03 than the family of generalized Gauss functions.

The methods and functions described in this article arent't entirely new. But the degree and extent of similarity to related work isn't quite clear. Hence embedding the stray light analysis of Junocam into related fields of research and results, together with the according bibliography, would be desirable.

A Parameters of Periodic Gauss for EFB03

The table headlines use G, μ, and σ without the tilde, but actually refer to the parameters of the periodic version of the Gauss functions. The meaning of the function F is that of the mean brightness functions at the lower and upper bound of the substripes, as in part I.

A.1 Substripe 0

Approximations $a_{x_1,x_2,0,1}G(\mu_{x_1,x_2,0,1}, \sigma_{x_1,x_2,0,1})$ of $F_{x_1,x_2,0,1}$
and $a_{x_1,x_2,0,2}G(\mu_{x_1,x_2,0,2}, \sigma_{x_1,x_2,0,2})$ of $F_{x_1,x_2,0,2}$ by sinus-modified Gauss functions

x_1	x_2	$a_{x_1,x_2,0,1}$	$\mu_{x_1,x_2,0,1}$	$\sigma_{x_1,x_2,0,1}$	$a_{x_1,x_2,0,2}$	$\mu_{x_1,x_2,0,2}$	$\sigma_{x_1,x_2,0,2}$
24	124	18.012032	20.803717	3.912027	18.522949	21.369310	3.832799
74	174	16.736149	20.984796	4.037648	17.258174	21.438157	3.904028
124	224	15.302618	21.119780	4.128121	15.833248	21.495926	3.984376
174	274	13.981488	21.227522	4.279986	14.665566	21.527415	4.140623
224	324	13.154696	21.283574	4.393807	14.004696	21.523294	4.296003
274	374	12.978897	21.290817	4.484056	13.911148	21.510270	4.386931
324	424	13.218515	21.285711	4.556601	14.283032	21.495290	4.443743
374	474	13.587911	21.292598	4.562738	14.643586	21.504662	4.447107
424	524	13.477408	21.337696	4.569187	14.324982	21.529712	4.407565
474	574	12.980298	21.424841	4.526092	13.577943	21.597304	4.395125
524	624	12.704641	21.535628	4.492881	13.195847	21.619522	4.414057
574	674	12.906374	21.602578	4.470545	13.445564	21.652710	4.414211
624	724	13.423945	21.643810	4.430454	14.078265	21.683934	4.380224
674	774	14.077516	21.668463	4.393974	14.889062	21.694970	4.336800
724	824	14.848972	21.705736	4.358432	15.749921	21.711467	4.288558
774	874	15.700220	21.749565	4.322581	16.880769	21.745861	4.233922
824	924	16.554592	21.807232	4.282880	18.172061	21.764235	4.182314
874	974	17.595246	21.850486	4.237727	19.397391	21.806787	4.111143
924	1024	19.167153	21.914582	4.186127	21.360412	21.857223	3.991434
974	1074	21.820107	22.081207	4.078018	24.428187	21.941115	3.838174

Approximations $a_{x_1,x_2,0,1}G(\mu_{x_1,x_2,0,1}, \sigma_{x_1,x_2,0,1})$ of $F_{x_1,x_2,0,1}$
and $a_{x_1,x_2,0,2}G(\mu_{x_1,x_2,0,2}, \sigma_{x_1,x_2,0,2})$ of $F_{x_1,x_2,0,2}$ by sinus-modified Gauss functions

x_1	x_2	$a_{x_1,x_2,0,1}$	$\mu_{x_1,x_2,0,1}$	$\sigma_{x_1,x_2,0,1}$	$a_{x_1,x_2,0,2}$	$\mu_{x_1,x_2,0,2}$	$\sigma_{x_1,x_2,0,2}$
1024	1124	25.732450	22.125535	3.968854	28.281724	21.984038	3.703104
1074	1174	29.512560	22.165582	3.807502	32.088212	21.983388	3.655412
1124	1224	31.771919	22.256012	3.640315	34.207328	21.991162	3.622183
1174	1274	33.232583	22.264194	3.580471	35.078944	22.009196	3.578780
1224	1324	34.801074	22.220986	3.590220	36.251529	22.029738	3.623948
1274	1374	38.380201	22.284858	3.544014	39.850070	22.060169	3.551063
1324	1424	46.832039	22.390863	3.270490	49.207019	22.114408	3.235031
1374	1474	65.304820	22.770319	2.902732	68.971450	22.259000	2.864453
1424	1524	85.830342	22.952959	2.919524	94.143121	22.351086	2.889916
1474	1574	82.075037	22.838715	3.073998	90.991996	22.339283	3.002448
1524	1624	83.984837	22.723318	2.802277	82.585559	22.216996	2.903288

A.2 Substripe 2

Approximations $a_{x_1,x_2,2,1}G(\mu_{x_1,x_2,2,1},\sigma_{x_1,x_2,2,1})$ of $F_{x_1,x_2,2,1}$
and $a_{x_1,x_2,2,2}G(\mu_{x_1,x_2,2,2},\sigma_{x_1,x_2,2,2})$ of $F_{x_1,x_2,2,2}$ by sinus-modified Gauss functions

x_1	x_2	$a_{x_1,x_2,2,1}$	$\mu_{x_1,x_2,2,1}$	$\sigma_{x_1,x_2,2,1}$	$a_{x_1,x_2,2,2}$	$\mu_{x_1,x_2,2,2}$	$\sigma_{x_1,x_2,2,2}$
24	124	16.366029	21.390684	3.890258	19.102794	21.853796	3.489733
74	174	15.405113	21.464990	3.917070	18.016875	21.861557	3.534811
124	224	14.198107	21.495415	4.024748	16.295627	21.850420	3.618297
174	274	13.237066	21.525281	4.152701	14.675980	21.789354	3.753119
224	324	12.729635	21.526195	4.265416	14.116149	21.791238	3.852790
274	374	12.623034	21.514007	4.349095	14.469065	21.798924	3.882396
324	424	12.830524	21.504716	4.425052	14.747377	21.797471	3.936151
374	474	13.065851	21.502285	4.450917	14.661902	21.821627	3.986507
424	524	12.837436	21.533950	4.395533	14.227464	21.819894	3.977264
474	574	12.363589	21.589101	4.335253	13.267214	21.796831	3.981223
524	624	12.108023	21.622343	4.339392	12.425935	21.777881	4.057794
574	674	12.195063	21.632595	4.373476	12.111812	21.748449	4.138050
624	724	12.710425	21.651940	4.366376	12.133633	21.734515	4.228459
674	774	13.376158	21.687915	4.361316	12.538438	21.722438	4.240178
724	824	14.084874	21.708382	4.336062	13.129192	21.712330	4.199112
774	874	15.074804	21.733752	4.285080	13.861004	21.712463	4.190113
824	924	16.222312	21.740654	4.227856	14.658317	21.737941	4.170401
874	974	17.314721	21.737865	4.159135	15.551418	21.743366	4.102995
924	1024	18.938443	21.791930	4.044618	16.930185	21.760425	4.004044
974	1074	21.634377	21.858856	3.886961	19.050550	21.754579	3.866472

Approximations $a_{x_1,x_2,2,1}G(\mu_{x_1,x_2,2,1},\sigma_{x_1,x_2,2,1})$ of $F_{x_1,x_2,2,1}$
and $a_{x_1,x_2,2,2}G(\mu_{x_1,x_2,2,2},\sigma_{x_1,x_2,2,2})$ of $F_{x_1,x_2,2,2}$ by sinus-modified Gauss functions

x_1	x_2	$a_{x_1,x_2,2,1}$	$\mu_{x_1,x_2,2,1}$	$\sigma_{x_1,x_2,2,1}$	$a_{x_1,x_2,2,2}$	$\mu_{x_1,x_2,2,2}$	$\sigma_{x_1,x_2,2,2}$
1024	1124	25.022454	21.920469	3.724092	21.878863	21.754252	3.710234
1074	1174	28.703088	21.932157	3.624126	25.566070	21.758577	3.501512
1124	1224	31.346401	21.962661	3.516069	29.411073	21.772964	3.277719
1174	1274	32.573792	21.973831	3.449535	30.975884	21.792657	3.202513
1224	1324	33.720552	21.981495	3.462668	30.486734	21.827116	3.245090
1274	1374	36.628545	22.026617	3.426666	31.472417	21.862977	3.193989
1324	1424	44.329746	22.084543	3.204653	36.080118	21.889208	3.039781
1374	1474	60.051436	22.169641	2.945431	45.595380	21.899292	2.939259
1424	1524	81.726330	22.234044	2.959232	56.446338	21.900635	3.112125
1474	1574	79.558103	22.239255	3.059057	57.479419	21.910762	3.122665
1524	1624	72.684206	22.181977	2.958535	59.134611	21.927101	2.875787

B Parameters of Power-Law Function $\hat{h}$ for EFB03

The tables of this section refer to parameters of the function $\hat{h}$ in the sense of equation (141). The parameter ϱ is chosen as the constant $\varrho := 79.0$. The column *RMS* lists the absolute root square error sum of the fitted approximation with respect to the data.

The column *it* lists the number of iteration steps needed to find the fit; it is provided as an informal hint to the computational complexity, since instead of the square error sum, the optimization was started with an optimization of a high power (exponents between 64 and 128) of the square error sum, to extend the zone of convergence. The Newton method converges much slower for this type of functions. The exponent was gradually reduced, whenever the changes of the error function became small, or when some predefined maximum number of iterations was exceeded. Only the last optimization steps were applied to the initial square error sum, where the Newton method converges fast.

B.1 Substripe 0, Lower Bound, $u_1 := 1.0$, $\zeta := 1.0$

Approximations $\hat{h}_x(u_1, u_2, \zeta, a, \alpha, \beta, \varrho, \lambda)$ of $F_{x_1,x_2,0,1}$, with $x_2 = x_1 + 100$, by power-law functions with $u_2 := \frac{1}{u_1}$

x_1	a	α	β	λ	σ	u_1	ζ	RMS	it
24	17.91180923	1.95686524	3.16230256	20.82305628	9.48909941	1.0	1.0	0.160465	458
74	16.73111052	1.93756798	3.10948232	20.94685755	9.68160454	1.0	1.0	0.164405	509
124	15.24481188	2.01884716	2.88265254	21.05364905	9.29721531	1.0	1.0	0.182035	332
174	13.84452614	2.21173673	2.27060299	21.15322358	8.04268093	1.0	1.0	0.196792	328
224	12.98634666	2.42578068	1.79840727	21.21767816	7.08468441	1.0	1.0	0.208243	326
274	12.81806385	2.56663637	1.55008259	21.24337775	6.62995224	1.0	1.0	0.222104	307
324	13.10413719	2.54182025	1.49529208	21.25328757	6.61233756	1.0	1.0	0.248739	265
374	13.53889187	2.36315639	1.62153306	21.26181880	7.01672138	1.0	1.0	0.285935	209
424	13.45776112	2.34850856	1.56236614	21.29924956	6.88616796	1.0	1.0	0.305290	183
474	12.84724451	2.53213489	1.35025631	21.39745325	6.26573480	1.0	1.0	0.297223	194
524	12.52102259	2.68745102	1.23663621	21.51866987	5.91910213	1.0	1.0	0.295331	206
574	12.69654323	2.73245641	1.18673188	21.59569969	5.77105868	1.0	1.0	0.307993	200
624	13.20075002	2.67733466	1.20194612	21.63804138	5.76815535	1.0	1.0	0.323207	183
674	13.86554408	2.57494666	1.25848852	21.66484762	5.87147142	1.0	1.0	0.339982	167
724	14.63894200	2.49005234	1.30784211	21.70370029	5.96187452	1.0	1.0	0.362104	151
774	15.50175520	2.45871735	1.31792270	21.75132437	5.93556687	1.0	1.0	0.385465	138
824	16.36116108	2.44706412	1.31463914	21.79909820	5.86997512	1.0	1.0	0.409916	127
874	17.38665470	2.43721738	1.30296115	21.85443263	5.78409607	1.0	1.0	0.446077	114
924	19.04699805	2.39389139	1.30882684	21.96227058	5.70321839	1.0	1.0	0.495876	97
974	21.71639742	2.37830828	1.19011651	22.11522986	5.24999505	1.0	1.0	0.563573	67
1024	25.53809588	2.38629788	1.05765923	22.20474516	4.76183935	1.0	1.0	0.709674	38
1074	29.24328805	2.36377429	1.07031177	22.25680402	4.62090706	1.0	1.0	0.867442	36
1124	31.47283308	2.52878705	0.98268512	22.31734581	4.21515763	1.0	1.0	0.912775	41
1174	32.86039460	2.92503049	0.80316972	22.32984495	3.74169460	1.0	1.0	0.914671	52
1224	34.32910101	3.38047917	0.67656762	22.30533150	3.52542116	1.0	1.0	0.924009	78
1274	37.19353288	3.58107355	0.64527010	22.32157109	3.46205596	1.0	1.0	0.977294	98
1324	44.79752193	3.39783030	0.69320028	22.49217956	3.30212228	1.0	1.0	1.272426	80
1374	62.39319734	3.51790712	0.67214093	22.86559440	2.90482364	1.0	1.0	2.002276	41
1424	80.93464054	3.92267251	0.60748488	23.05715460	2.84990105	1.0	1.0	2.607743	51
1474	79.05187650	3.55030417	0.69830909	22.92079268	3.08245922	1.0	1.0	2.377627	55
1524	80.16369760	2.77964210	0.97718688	22.76288374	3.28519021	1.0	1.0	2.051840	46

B.2 Substripe 0, Lower Bound, u_1 Optimized, $\zeta := 1.0$

Approximations $\hat{h}_x(u_1, u_2, \zeta, a, \alpha, \beta, \varrho, \lambda)$ of $F_{x_1,x_2,0,1}$, with $x_2 = x_1 + 100$, by power-law functions with $u_2 := \frac{1}{u_1}$

x_1	a	α	β	λ	σ	u_1	ζ	RMS	it
24	17.96246578	1.92299860	3.38523052	21.07252572	9.95201545	1.06046466	1.0	0.118030	550
74	16.74557610	1.92092265	3.22642134	21.13344653	9.94209124	1.04395826	1.0	0.146125	696
124	15.24372546	2.00763823	2.96234503	21.26937405	9.47607941	1.04912989	1.0	0.165248	391
174	13.83396552	2.20093923	2.32721908	21.47587209	8.16997797	1.07109050	1.0	0.171090	383
224	12.97989541	2.40281065	1.86298851	21.65446111	7.22681577	1.09418114	1.0	0.172215	386
274	12.82213153	2.52190941	1.63455233	21.77432651	6.81397929	1.11307102	1.0	0.175197	382
324	13.13550164	2.45279501	1.65039460	21.88117693	6.96298382	1.13431444	1.0	0.184192	359
374	13.58971949	2.25226168	1.87292956	21.94327111	7.62587581	1.14806909	1.0	0.208581	303
424	13.51459559	2.22292539	1.83638797	22.00174221	7.57142677	1.15382973	1.0	0.228274	268
474	12.89402136	2.40183143	1.54932238	22.13199720	6.73821666	1.16113851	1.0	0.223315	277
524	12.56954386	2.54330416	1.41085787	22.31836316	6.30116000	1.17687163	1.0	0.214360	300
574	12.75608081	2.56974208	1.36604177	22.44129437	6.15314816	1.18890655	1.0	0.217747	302
624	13.26567576	2.51562547	1.38786872	22.47977413	6.16901398	1.19030952	1.0	0.227792	285
674	13.93613877	2.42023946	1.45819355	22.47774938	6.31648033	1.18582266	1.0	0.242322	265
724	14.72483375	2.32943223	1.53758437	22.50602951	6.48891797	1.18582863	1.0	0.257456	249
774	15.60201045	2.29187717	1.56184929	22.55775482	6.49605708	1.18952912	1.0	0.269858	241
824	16.47320248	2.27659469	1.56341831	22.62617256	6.43359654	1.19750795	1.0	0.278004	242
874	17.51658572	2.25880485	1.56266500	22.72711970	6.36103773	1.21263059	1.0	0.286806	250
924	19.22480818	2.19096433	1.61679105	22.86033834	6.39097636	1.22565313	1.0	0.298664	272
974	21.96756603	2.14217376	1.50473705	23.01332235	5.95935692	1.23666726	1.0	0.307698	364
1024	25.92079556	2.08621501	1.40696820	23.15178098	5.55361554	1.26433675	1.0	0.357469	405
1074	29.76504607	2.02201911	1.49693573	23.23600701	5.56799488	1.28800259	1.0	0.440120	348
1124	31.96814951	2.19620658	1.29269465	23.28544904	4.79894045	1.29600848	1.0	0.462308	364
1174	33.25421560	2.59228637	0.98227472	23.30793667	3.98766881	1.30279462	1.0	0.459575	423
1224	34.61035125	3.06738679	0.78139289	23.29764172	3.60975242	1.30213908	1.0	0.483892	435
1274	37.57638538	3.18856185	0.75987391	23.31075577	3.54789177	1.29998617	1.0	0.521697	437
1324	46.17857676	2.64571038	1.00255966	23.52872001	3.64008178	1.34803891	1.0	0.555103	524
1374	64.82698876	2.63917393	1.02801479	24.00114259	3.17486845	1.45492626	1.0	0.675289	191
1424	82.96906969	3.16704719	0.81384033	24.25674634	2.89449531	1.47341955	1.0	1.092960	167
1474	80.47572336	3.04301186	0.86780496	23.96289277	3.16329480	1.37847201	1.0	1.261321	139
1524	81.21077894	2.53323642	1.13057949	23.56559637	3.40519857	1.30582568	1.0	1.078308	133

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